ASYMMETRIC SEQUENTIAL SEARCH UNDER INCOMPLETE INFORMATION

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Asymmetric Sequential Search under Incomplete Information

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Abstract

We study a two-stage sequential search model with two agents who compete for one job. The agents arrive sequentially, each one in a different stage. The agents' abilities are private information and they are derived from heterogeneous distribution functions. In each stage the designer chooses an ability threshold. If an agent has a higher ability than the ability threshold in the stage in which he arrives, he gets the job and the search is over. We analyze the equilibrium ability thresholds imposed by the designer who wishes to maximize the ability of the agent who gets the job minus the search cost. We also investigate the ratio of the equilibrium ability thresholds as well as the optimal allocation of agents in both stages according to the agents' distributions of abilities.

1 Introduction

The problem of assigning various items of two different groups which are not necessarily of the same size such as a group of agents to a group of jobs have been studied in numerous frameworks. In his classical model, Derman et al. (1972) considered n agents who are available to perform n jobs which arrive in a sequential order. The designer has to decide whether to assign the job at all, and if so, which of the n

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agents to assign it to. However, an agent can be assigned to a job only once. The goal of the designer is to maximize the expected total return, where the return from an allocation of job j to agent k is the product of the value of the job j and the value (ability) of the agent k. It is assumed that the agents' values are commonly known but the jobs' values are random variables. Later, Albright (1974) focused on a continuous time framework where the number of agents is fixed but the jobs arrive randomly according to a continuous scholastic process. In the latter two models the agents' values are common knowledge. Mussa and Rosen (1978), on the other hand, considered a static model of a monopolistic seller who decides on a price function that depends on the quality of the product for sale when he does not know the agents' values for these products, but does know their distribution. Similarly, Segal (2003) studied a monopolistic seller who faces n buyers, each of whom has unit demand and private information for the unit being sold, and the seller does not know the buyers' valuations nor their distribution. A dynamic model where the agents arrive by a stochastic process was studied by Gershkov and Moldovanu (2009a). Using mechanism design tools, they derived a revenue maximization policy in continuous time frameworks under the assumptions that agents' valuations are drawn by a distribution function that is known to the seller and is identical for all agents. In later works, Gershkov and Moldovanu (2009b, 2012) applied a discrete time framework where they assumed that the agents' values are private information and even their distributions are unknown to the designer.

In all the above assignment models, as well as various others in the literature, it was assumed that the agents' types under incomplete information, whether their distribution is known or unknown, are symmetric; namely, each type is derived from the same distribution function. In this paper we study a very simple model of assignment in which the agents' types are under incomplete information and are not derived from the same distribution function. To put it formally, in the model, two agents compete for one job. The agents arrive sequentially one after the other. The designer wishes to give the job to the agent with the highest ability. He does not know the agents' abilities, but does know their distribution functions which are heterogeneous. In the first stage, the designer decides an ability threshold. Then, the first agent arrives and if his ability is higher than or equal to the ability threshold imposed by the designer he wins the job and the sequential search is over. Otherwise, in the second stage the designer again imposes an ability threshold which is not necessarily equal to the previous one. Then, the second agent arrives, and if his ability is higher than or equal to the second ability threshold, he wins the job. If, on the other hand, the second agent's ability is lower than the designer's ability threshold, no one wins the job and the payoff of the designer is negative and equals $-2c$ where c is his cost of the search in each stage. The goal of the designer is to maximize his expected payoff which is equal to a monotonic function of the expected ability of the agent who gets the job minus his cost of the search.

We begin the analysis by examining the effect of time on the designer's strategies; namely, we investigate the ratio of the ability thresholds imposed by the designer in the two stages of the sequential search. We particularly address the following questions: Is it true that these ability thresholds decrease along the stages of the sequential search? How does the asymmetry of the agents affect the ratio of the ability thresholds they need to face? In order to answer these questions we say that agent 1 is stronger than agent 2 if the distribution of agent 1's ability stochastically dominates his opponent's distribution of ability in terms of the hazard rate. Then we show that, if the agent who arrives in the first stage is stronger than the agent who arrives in the second stage, regardless of the value of the designer's search cost c, in the perfect Bayesian equilibrium the stronger agent in the first period faces a higher ability threshold than his opponent in the second stage. However, in the opposite case when the stronger agent arrives in the second stage, if the designer's cost c is sufficiently high then the stronger agent in the second stage may face a lower ability threshold than his opponent. If, on the other hand, the search cost c is sufficiently small, although the stronger agent arrives in the second stage, his weaker opponent in the first stage faces a higher ability threshold.

We also compare the equilibrium ability thresholds in two different sequential search models. We show that if the agent in each stage of the sequential search A is stronger than the agent in the same stage of the sequential search B, then the equilibrium ability threshold in each stage of the sequential search A is higher than the equilibrium ability threshold in the same stage of the sequential search B. We also show that if the agent in each stage of the sequential search A is stronger than the agent in the same stage of the sequential search B, and also that the stronger agent in each sequential search arrives in the first stage, then the designer's expected payoff in the sequential search A is larger than in the sequential search B.

We then deal with the question of what the optimal order of agents is for the designer who wishes to maximize his expected payoff which is equal to the expected ability of the agent who gets the job minus the cost of the search. We first show that in our sequential search model if the designer has to impose the same ability threshold in both stages then the stronger player should be allocated in the first stage. In the case that the designer is allowed to impose different ability thresholds in both stages of the sequential search, we provide sufficient conditions such that the stronger player should be allocated in the first stage of the sequential search.

The literature of economic theory under incomplete information has dealt mostly with symmetric agents where their types are derived from the same distribution function which is common knowledge. In many cases, however, agents' types are drawn from different distribution functions which makes it hard to deal with these models particularly when the agents act simultaneously. Although explicit expressions for such asymmetric equilibrium strategies cannot be obtained other than for very simple models, we can find some works which dealing with asymmetric auctions or contests (see, for example, Amman and Leininger (1996), Maskin and Riley (2000), Fibich, Gavious and Sela (2004), Parreiras and Rubinchik (2010), Kirkegaard (2012) and Gavious and Minchuk (2014)). In contrast, when the agents act sequentially and not simultaneously, the analysis of asymmetric models under incomplete information becomes more tractable, in particular when each agent plays only in one stage like in Segev and Sela (2014) who study asymmetric sequential all-pay auctions under incomplete information. Since in our sequential search model each agent similarly plays at most in one stage it is also tractable even though the players are ex-ante asymmetric.

2 The model

We consider a two-stage sequential search with two agents who compete to win one job. The agents' valuation for winning the job is normalized to 1. We denote by $a_i \geq 0$ the ability (or type) of agent $i, i = 1, 2$ which is private information to i. Agent i's ability is independently drawn from the interval $[0, 1]$ according to a distribution function F_i which is common knowledge. We assume that F_i has a positive and continuous density function $F_i' > 0$. Agent i participates in stage i only. In the first stage, the designer imposes an ability threshold d_1 and then agent 1 arrives. If his ability is higher than or equal to d_1 he wins the job. If, on the other hand, agent 1's ability is smaller than d_1, the designer imposes a new ability threshold d_2 in the second stage. Then, agent 2 arrives, and if his ability is higher than or equal to d_2 he wins the job;

otherwise, the search ends and no one wins the job. The contest designer has a search cost of $c \geq 0$ in each stage. He wishes to maximize his expected payoff which is equal to $g(d_i) - \sum_{j=1}^{i} c_j$ where i is the stage in which agent i wins the job. The function $g(d_i)$ could be equal to the average agent's ability to win the job, i.e., $g(d_i) = \frac{\int_{d_i}^{1} a_i F_i'(a)da}{1-F_i(d_i)}$ or to any other monotonic increasing function of d_i. In the case that no agent wins the job, the designer's expected payoff is $-\sum_{j=1}^{2} c_j$.

3 The equilibrium ability thresholds

In order to analyze the perfect Bayesian equilibrium of the model, we begin with the second stage of the sequential search and go backwards to the previous stage. The designer's maximization problem in stage 2 is

$$\max_{d_2} g(d_2)(1 - F_2(d_2)) - c \tag{1}$$

Thus, the equilibrium ability threshold in stage 2, d_2^*, is given by

$$g'(d_2^*)(1 - F_2(d_2^*)) - g(d_2^*)F_2'(d_2^*) = 0 \tag{2}$$

The designer's maximization problem in stage 1 is then

$$\max_{d_1} g(d_1)(1 - F_1(d_1)) + F_1(d_1)g(d_2^*)(1 - F_2(d_2^*)) - c(1 + F_1(d_1)) \tag{3}$$

where d_2^* is the equilibrium ability threshold in stage 2. The equilibrium ability threshold in stage 1, d_1^* is then given by

$$g'(d_1^*)(1 - F_1(d_1^*)) - g(d_1^*)F_1'(d_1^*) + F_1'(d_1^*)g(d_2^*)(1 - F_2(d_2^*)) - cF_1'(d_1^*) = 0 \tag{4}$$

We assume that the maximization problems (3) and (1) have a solution.[1] The solution (d_1^*, d_2^*) of equations (4) and (2) provides the equilibrium ability thresholds imposed by the designer. Note that these equations are the necessary conditions for the optimal ability threshold, and therefore the optimal ability thresholds are also equilibrium ability thresholds. The following example illustrates the perfect Bayesian equilibrium in our sequential search model.

[1] For example, a sufficient condition that the maximization problems (1) and (3) have a solution is that $g(\cdot)$ is a concave function and $F_i, i = 1, 2$ are convex functions.

Example 1 *Suppose that the players' abilities are distributed according to $F_1(a) = F_2(a) = a$ and $g(d) = d$. Then, the designers's maximization problem in the second stage is*

$$\max_{d_2} d_2(1 - d_2) - c$$

By (2), the F.O.C. is

$$1 - 2d_2^* = 0 \implies d_2^* = 0.5$$

The designer's maximization problem in the first stage is

$$\max_{d_1} d_1(1 - d_1) + d_1 d_2^*(1 - d_2^*) - c(1 + d_1)$$

By (4), the F.O.C. is

$$\begin{aligned} 1 - 2d_1^* + (0.25 - c) &= 0 \\ &\implies d_1^* = 0.625 - 0.5c \end{aligned}$$

The designer's expected payoff is given by

$$\begin{aligned} E_d &= (\frac{5}{8} - \frac{1}{2}c)(1 - (\frac{5}{8} - \frac{1}{2}c)) + (\frac{5}{8} - \frac{1}{2}c)\frac{1}{4} - c(1 + \frac{5}{8} - \frac{1}{2}c) \\ &= \frac{1}{4}c^2 - \frac{13}{8}c + \frac{25}{64} \end{aligned}$$

The following figure presents the designer's expected payoff.

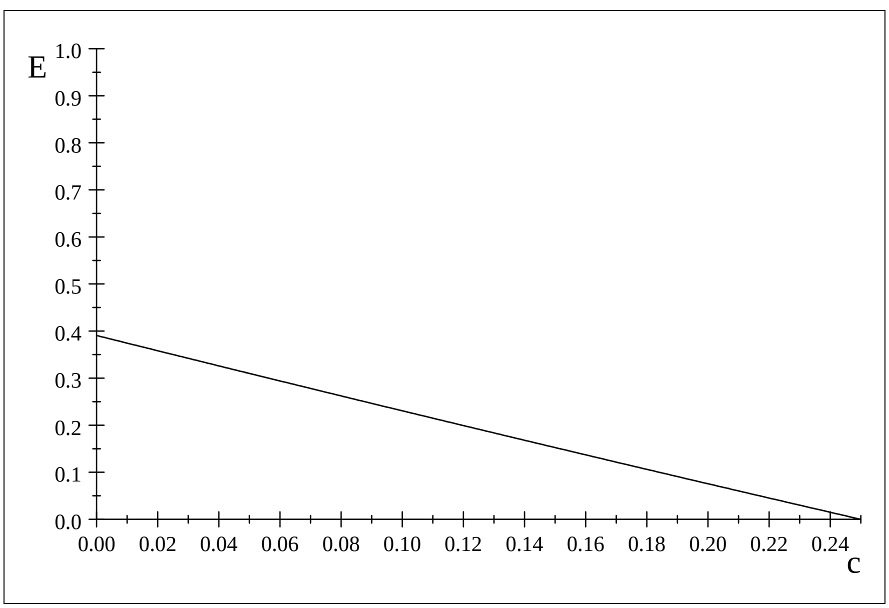

Figure 1: The designer's expected payoff as a function of the search cost c.

We can see that for all $c < \frac{1}{4}$ the equilibrium ability threshold d_1^* in the first stage is higher than or equal to the equilibrium ability threshold d_2^* in the second stage. For $c = \frac{1}{4}$ the equilibrium ability thresholds are the same and both are equal to $\frac{1}{2}$ where the expected payoff of the designer is zero. For $c > \frac{1}{4}$ the designer has a negative expected payoff. Thus, for every possible value of the cost $c \leq 0.25$ we obtain that $d_1^* \geq d_{2.}^*$.

In order to proceed we need the following definition. Let F and G be two distribution functions with hazard rates $\lambda_F = \frac{F'(x)}{1-F(x)}$ and $\lambda_G = \frac{G'(x)}{1-G(x)}$, respectively. If for all x, $\lambda_F(x) \leq \lambda_G(x)$, we say that F stochastically dominates G in terms of the hazard rate. Then, if player i's ability is distributed according to

F_i, $i = 1, 2$, and F_1 stochastically dominates F_2 in terms of the hazard rate, we say that agent 1 is stronger than agent 2.

The following result demonstrates that if the agent who arrives in the first stage is stronger than the agent who arrives in the second stage, then the stronger agent in the first stage has to face a higher ability threshold than the weaker agent in the second stage.

Proposition 1 *If F_1 stochastically dominates F_2 in terms of the hazard rate, then regardless of the value of the designer's search cost c, the equilibrium ability threshold in the first stage d_1^* is higher than or equal to the equilibrium ability threshold in the second stage d_2^*.*

Proof. The equilibrium ability threshold in the second stage is determined by

$$g'(d_2^*)(1 - F_2(d_2^*)) - g(d_2^*)F_2'(d_2^*) = 0 \tag{5}$$

while the equilibrium effort threshold in the first stage is determined by

$$G(d_1^*) = g'(d_1^*)(1 - F_1(d_1^*)) - g(d_1^*)F_1'(d_1^*) + F_1'(d_1^*)g(d_2^*)(1 - F_2(d_2^*)) - cF_1'(d_1^*) = 0 \tag{6}$$

Inserting (5) in (6) yields

$$\begin{aligned} G(d_1^* \quad &: \quad d_1^* = d_2^*) = G(d_2^*) = g'(d_2^*)(1 - F_1(d_2^*)) - g(d_2^*)F_1'(d_2^*)F_2(d_2^*) - cF_1'(d_2^*) \qquad (7) \\ &= \quad g(d_2^*)\frac{F_2'(d_2^*)(1 - F_1(d_2^*))}{(1 - F_2(d_2^*))} - g(d_2^*)F_1'(d_2^*)F_2(d_2^*) - cF_1'(d_2^*) \\ &= \quad F_1'(d_2^*)\left[g(d_2^*)\frac{F_2'(d_2^*)(1 - F_1(d_2^*))}{(1 - F_2(d_2^*))F_1'(d_2^*)} - g(d_2^*)F_2(d_2^*) - c\right] \\ &= \quad F_1'(d_2^*)\left[g(d_2^*)\frac{\lambda_{F_2}(d_2^*)}{\lambda_{F_1}(d_2^*)} - g(d_2^*)F_2(d_2^*) - c\right] \end{aligned}$$

where $\lambda_{F_1}(a)$ and $\lambda_{F_2}(a)$ are the hazard rates of F_1 and F_2, respectively. We assume that F_1 stochastically dominates F_2 in terms of the hazard rate which implies that $\frac{\lambda_{F_2}(d_2^*)}{\lambda_{F_1}(d_2^*)} \geq 1$ and therefore we have that

$$G(d_2^*) \geq F_1'(d_2^*)\left[g(d_2^*)(1 - F_2(d_2^*)) - c\right] \geq 0$$

The last inequality holds since otherwise the designer has no incentive to act in the second stage. Thus, $G(d_2^*) \geq 0$, and since $G(d_1^*) = 0$ and $G' < 0$, we obtain that the equilibrium ability thresholds satisfy $d_1^* \geq d_2^*$.

■

The result of Proposition 1 implies that if both agents have the same distribution of ability, i.e., $F_1 = F_2$, the equilibrium ability threshold in the first stage d_1^* is higher than or equal to the equilibrium ability threshold in the second stage d_2^*. The next result shows that when agent 1 is not stronger than agent 2, the equilibrium ability threshold that the weaker agent (agent 1) faces in the first stage might be either higher or lower than the equilibrium ability threshold that the stronger agent (agent 2) faces in the second stage.

Proposition 2 *Assume that F_2 stochastically dominates F_1 in terms of the hazard rate. Then, if the cost c is sufficiently high, the equilibrium ability threshold in the first stage d_1^* is lower than the equilibrium ability threshold in the second stage d_2^*. On the other hand, if the cost c is sufficiently small and $\frac{\lambda_{F_2}(a)}{\lambda_{F_1}(a)} > F_2(a)$ for all a, then the equilibrium ability threshold in the first stage d_1^* is higher than the equilibrium ability threshold in the second stage d_2^*.*

Proof. By (6) and (7) the equilibrium ability thresholds satisfy

$$G(d_1^*) = g'(d_1^*)(1 - F_1(d_1^*)) - g(d_1^*)F_1'(d_1^*) + F_1'(d_1^*)g(d_2^*)(1 - F_2(d_2^*)) - cF_1'(d_1^*) = 0$$

and

$$G(d_2^*) = F_1'(d_2^*)\left[g(d_2^*)\frac{\lambda_{F_2}(d_2^*)}{\lambda_{F_1}(d_2^*)} - g(d_2^*)F_2(d_2^*) - c\right]$$

Since $\frac{\lambda_{F_2}(a)}{\lambda_{F_1}(a)} < 1$, if the term $g(d_2^*)(\ 1 - F_2(d_2^*)) - c$ is sufficiently small we obtain that $G(d_2^*) < 0$, and since $G' < 0$ we have that $d_1^* < d_2^*$. On the other hand, if $\frac{\lambda_{F_2}(a)}{\lambda_{F_1}(a)} \geq F_2(a)$ and c is sufficiently small, we obtain that $G(d_2^*) > 0$ which implies that $d_1^* > d_2^*$. ■

The following example illustrates the result obtained in Proposition 2.

Example 2 *Suppose that $F_1(x) = x^k$, $F_2(x) = x^m$, $g(d) = d$ and $c = 0$. Let $k \leq m$ such that F_2 stochastically dominates F_1 in terms of the hazard rate. We will show that although the stronger agent arrives in the second stage, the equilibrium ability threshold in the first stage is higher than in the second stage. In that case we have that*

$$\frac{\lambda_{F_2}(x)}{\lambda_{F_1}(x)} = \frac{\frac{F_2'(x)}{1-F_2(x)}}{\frac{F_1'(x)}{1-F_1(x)}} = \frac{\frac{mx^{m-1}}{1-x^m}}{\frac{kx^{k-1}}{1-x^k}} = \frac{mx^m(1-x^k)}{kx^k(1-x^m)}$$

By Proposition 2, in order to show that the equilibrium ability threshold in the second stage is higher than in the first stage, it is sufficient to show that for all $0 \leq x \leq 1$,

$$\frac{\lambda_{F_2}(x)}{\lambda_{F_1}(x)} = \frac{mx^m(1-x^k)}{kx^k(1-x^m)} \geq x^m = F_2(x)$$

or, alternatively, we need to show that

$$m + kx^{k+m} \geq (k+m)x^k$$

For $x = 1$, both sides of the last inequality are the same. The derivative of the LHS is

$$k(k+m)x^{k+m-1}$$

and the derivative of the RHS is

$$k(k+m)x^{k-1}$$

Since for every $0 \leq x \leq 1$ there exists $x^{k+m-1} < x^{k-1}$, we obtain that $m + kx^{k+m} \geq (k+m)x^k$ for every $0 \leq x \leq 1$.

Thus far we have compared the equilibrium ability thresholds of the same sequential search model. Now we wish to compare the equilibrium ability thresholds of two different sequential search models.

Proposition 3 *Let (d_1^F, d_2^F) be the equilibrium ability thresholds in the sequential search when the distribution functions are $F_i, i = 1, 2$, and (d_1^G, d_2^G) be the equilibrium ability thresholds in the sequential search when the distribution functions are $G_i, i = 1, 2$. If $F_i, i = 1, 2$ stochastically dominates $G_i, i = 1, 2$ in terms of the hazard rate, then, $d_j^F \geq d_j^G, j = 1, 2$.*

Proof. By (2) we have the following conditions

$$\begin{aligned} g'(d_2^F)\frac{(1-F_2(d_2^F))}{F_2'(d_2^F)} - g(d_2^F) &= 0 \\ g'(d_2^G)\frac{(1-G_2(d_2^G))}{G_2'(d_2^G)} - g(d_2^G) &= 0 \end{aligned}$$

Rearranging implies that

$$\begin{aligned} \frac{F_2'(d_2^F)}{(1-F_2(d_2^F))} &= \frac{g'(d_2^F)}{g(d_2^F)} \\ \frac{G_2'(d_2^G)}{(1-G_2(d_2^G))} &= \frac{g'(d_2^G)}{g(d_2^G)} \end{aligned}$$

Since F_2 stochastically dominates G_2 in terms of the hazard rate, $\frac{(1-F_2(x))}{F_2'(x)} \geq \frac{(1-G_2(x))}{G_2'(x)}$ and, in particular, we have that

$$L_F(d_2^G) = \frac{g'(d_2^G)(1-F_2(d_2^G))}{F_2'(d_2^G)} - g(d_2^G) \geq \frac{g'(d_2^G)(1-G_2(d_2^G))}{G_2'(d_2^G)} - g(d_2^G) = L_G(d_2^G) \tag{8}$$

By the S.O.C. of the maximization problem (1) we obtain that $L_F(d)$ and $L_G(d)$ are decreasing functions, and therefore by (8) in order to obtain the equality $L_F(d_2^F) = L_G(d_2^G)$ we necessarily have that $d_2^F > d_2^G$.

Similarly, by (4) we have the following conditions

$$\begin{aligned} g'(d_1^F)(1-F_1(d_1^F)) - g(d_1^F)F_1'(d_1^F)) + F_1'(d_1^F)g(d_2^F)(1-F_2(d_2^F)) &= 0 \\ g'(d_1^G)(1-G_1(d_1^G)) - g(d_1^G)G_1'(d_1^G) + G_1'(d_1^G)g(d_2^G)(1-G_2(d_2^G)) &= 0 \end{aligned}$$

Since

$$g(d_2^F)(1-F_2(d_2^F)) \geq g(d_2^G)(1-F_2(d_2^G)) \geq g(d_2^G)(1-G_2(d_2^G))$$

we obtain that

$$M_F(d_1^F) = \frac{g'(d_1^F)(1-F_1(d_1^F))}{F_1'(d_1^F)} - g(d_1^F) \leq \frac{g'(d_1^G)(1-G_1(d_1^G))}{G_1'(d_1^G)} - g(d_1^G) = M_G(d_1^G)$$

Since F_1 stochastically dominates G_1 in terms of the hazard rate, $\frac{(1-F_1(x))}{F_1'(x)} \geq \frac{(1-G_1(x))}{G_1'(x)}$ and, in particular, we have that

$$M_F(d_1^G) = \frac{g'(d_1^G)(1-F_1(d_1^G))}{F_1'(d_1^G)} - g(d_1^G) \geq \frac{g'(d_1^G)(1-G_1(d_1^G))}{G_1'(d_1^G)} - g(d_1^G) = M_G(d_1^G) \tag{9}$$

By the S.O.C. of the maximization problem (3) we obtain that $M_F(d)$ and $M_G(d)$ are decreasing functions, and therefore by (9) in order to obtain the equality $M_F(d_1^F) = M_G(d_1^G)$ we necessarily have that $d_1^F > d_1^G$.

■

In the following, using the above results concerning the ratio of the equilibrium ability thresholds, we show that if the agent in each stage of sequential search A is stronger than the agent at the same stage of sequential search B, and the stronger agent of the sequential search B arrives in the first stage, then the designer's optimal expected payoff in the sequential search A is higher than in the sequential search B.

Proposition 4 *Suppose that $F_i, i = 1, 2$ stochastically dominates $G_i, i = 1, 2$ in terms of the hazard rate, and G_1 stochastically dominates G_2 in terms of the hazard rate as well. Then, in the sequential search with the distribution functions $F_i, i = 1, 2$ the designer's optimal expected payoff is higher than in the sequential search with the distribution functions $G_i, i = 1, 2$.*

Proof. Let (d_1^F, d_2^F) be the optimal ability thresholds when the distribution functions are $F_i, i = 1, 2$, and (d_1^G, d_2^G) be the optimal ability thresholds when the distribution functions are $G_i, i = 1, 2$. We wish to show that

$$g(d_1^F)(1 - F_1(d_1^F)) + F_1(d_1^F)(g(d_2^F)(1 - F_2(d_2^F)) - c) \geq g(d_1^G)(1 - G_1(d_1^G)) + G_1(d_1^G)(g(d_2^G)(1 - G_2(d_2^G)) - c)$$

Since the optimal ability thresholds satisfy equations (4) and (2), by Proposition 1 we have that $d_1^G \geq d_2^G$. Thus, it is sufficient to show that for all $d_1 \geq d_2$

$$g(d_1)(1 - F_1(d_1)) + F_1(d_1)(g(d_2)(1 - F_2(d_2)) - c) \geq g(d_1)(1 - G_1(d_1)) + G_1(d_1)(g(d_2)(1 - G_2(d_2)) - c)$$

or, alternatively, that

$$(g(d_1) - g(d_2) + c)(G_1(d_1) - F_1(d_1)) \geq g(d_2)(-G_1(d_1)G_2(d_2) + F_1(d_1)F_2(d_2)) \tag{10}$$

Since $d_1 \geq d_2$ and F_i stochastically dominates G_i, we obtain that $g(d_1) - g(d_2) > 0$ and $G_1(d_1) - F_1(d_1) > 0$, and, in particular, that the LHS of (10) is always positive. On the other hand, $G_1(d_1)G_2(d_2) > F_1(d_1)F_2(d_2)$ and therefore the RHS of (10) is negative. Hence, the inequality (10) holds. ■

4 The optimal order of agents

In this section we examine how the order of agents affects the designer's expected payoff. Assume first that the contest designer has a limitation according to which the ability thresholds have to be the same in both stages although the agents have asymmetric distributions of abilities. Then, if the designer has to choose the same ability threshold independent of the order of agents we obtain that

Proposition 5 *If the designer imposes the same ability threshold in both stages and the distribution function of player 1's ability F_1 stochastically dominates the distribution function of player 2's ability F_2 in terms of the hazard rate, then the designer maximizes his expected payoff when player 1 is allocated in the first stage and player 2 in the second one.*

Proof. It is sufficient to show that for every $0 \leq d \leq 1$

$$h_1(d) = g(d)(1-F_1(d))+F_1(d)(g(d)(1-F_2(d))-c) \geq g(d)(1-F_2(d))+F_2(d)(g(d)(1-F_1(d))-c) = h_2(d) \quad (11)$$

Note that

$$h_1(d) - h_2(d) = c(F_2(d) - F_1(d))$$

Since $F_1(d) \leq F_2(d)$ for every $0 \leq d \leq 1$, the inequality (11) is satisfied. ■

For Proposition 5 it was sufficient to assume that F_1 first-order stochastically dominates F_2, i.e., for all x, $F_1(x) \leq F_2(x)$. Note that the fact that F_1 stochastically dominates F_2 in terms of the hazard rate implies that F_1 first-order stochastically dominates F_2. Assume now that the designer has to choose the same ability threshold independent of the order of agents where this ability threshold is equal to the equilibrium ability threshold of the second stage. Then, in contrast to Proposition 5, the following result shows that the stronger player should be allocated in the second stage.

Proposition 6 *Assume that the cost of search c is sufficiently small and the designer imposes in both stages the equilibrium ability threshold of the second stage. Then, if the distribution function of player 1's ability F_1 stochastically dominates the distribution function of player 2's ability F_2 in terms of the hazard rate, the designer maximizes his expected payoff when player 1 is allocated in the second stage and player 2 in the first one.*

Proof. Let d_1 be the ability threshold for both stages when player 1 is allocated in the second stage and d_2 be the ability threshold for both stages when player 2 is allocated in the second stage. We wish to show that

$$g(d_1)(1-F_2(d_1))+F_2(d_1)(g(d_1)(1-F_1(d_1))-c)\geq g(d_2)(1-F_1(d_2))+F_1(d_2)(g(d_2)(1-F_2(d_2))-c)$$

or, alternatively, that

$$g(d_1)(1-F_2(d_1)F_1(d_1))-g(d_2)(1-F_2(d_2)F_1(d_2))\geq(F_2(d_1)-F_1(d_2))c$$

Assume that $c=0$. Then we need to show that

$$G=g(d_1)(1-F_2(d_1)F_1(d_1))-g(d_2)(1-F_2(d_2)F_1(d_2))\geq 0$$

Let $h(d)=g(d)(1-F_2(d)F_1(d))$. Since $F_2(d)F_1(d)$ first-order stochastically dominates both F_1 and F_2, and F_1 stochastically dominates F_2 we obtain that

$$\arg\max g(d)(1-F_2F_1)>\arg\max g(d)(1-F_1)>\arg\max g(d)(1-F_2)$$

Thus, $d_1, d_2<\arg\max h(d)=g(d)(1-F_2(d)F_1(d))$ and since by Proposition 3 $d_1>d_2$, we obtain that

$$G=g(d_1)(1-F_2(d_1)F_1(d_1))-g(d_2)(1-F_2(d_2)F_1(d_2))\geq 0$$

■

For Proposition 6 it was also sufficient to assume that F_1 first-order stochastically dominates F_2. We now give the designer the freedom to choose different ability thresholds that depend on the agents' distributions of abilities. Then, we have sufficient conditions that the stronger player should be allocated in the first stage of the sequential search.

Proposition 7 *Assume that the distribution function of player 1's ability F_1 stochastically dominates the distribution function of player 2's ability F_2 in terms of the hazard rate. Then, if the search cost c is sufficiently small and*

$$\frac{\lambda_{F_1}(x)}{\lambda_{F_2}(x)}\geq F_1(x) \text{ for all } 0\leq x\leq 1 \tag{12}$$

the designer maximizes his expected payoff when he allocates player 1 in the first stage.

Proof. Let $(\widetilde{d}_1,\widetilde{d}_2)$ be the equilibrium ability thresholds when player 1 is allocated in the first stage, and $(\widehat{d}_1,\widehat{d}_2)$ be the equilibrium ability thresholds when player 2 is allocated in the first stage. We wish to show that in order to maximize his expected payoff the designer should allocate player 1 in the first stage and player 2 in the second one. Thus, we need to show that

$$g(\widetilde{d}_1)(1 - F_1(\widetilde{d}_1)) + F_1(\widetilde{d}_1)(g(\widetilde{d}_2)(1 - F_2(\widetilde{d}_2)) - c) \geq g(\widehat{d}_1)(1 - F_2(\widehat{d}_1)) + F_2(\widehat{d}_1)(g(\widehat{d}_2)(1 - F_1(\widehat{d}_2)) - c) \quad (13)$$

By Proposition 2, since $\frac{\lambda_{F_1}(x)}{\lambda_{F_2}(x)} \geq F_1(x)$ and c is sufficiently small, then $\widehat{d}_1 > \widehat{d}_2$. Thus, it is enough to show that for all $d_1 > d_2$,

$$g(d_1)(1 - F_1(d_1)) + F_1(d_1)(g(d_2)(1 - F_2(d_2)) - c) \geq g(d_1)(1 - F_2(d_1)) + F_2(d_1)(g(d_2)(1 - F_1(d_2)) - c)$$

or, alternatively, that

$$\begin{aligned} & g(d_1)(F_2(d_1) - F_1(d_1)) - (g(d_2) - c)(F_2(d_1) - F_1(d_1)) \\ = \quad & (g(d_1) - g(d_2) + c)(F_2(d_1) - F_1(d_1)) \geq 0 \end{aligned}$$

Since F_1 stochastically dominates F_2 in terms of the hazard rate the last inequality holds. ■

The result of Proposition 7 is illustrated in the following example.

Example 3 *Suppose that* $F_1(x) = x^k, F_2(x) = x^m, k > m \geq 1$, *and* $c = 0$. *Then, by Proposition 7, player 1 should be allocated in the first stage if*

$$\frac{F_1'(x)(1 - F_2(x))}{F_2'(x))(1 - F_1(x))} \geq F_1(x)$$

This condition can be expressed as

$$\frac{kx^{k-1}(1 - x^m)}{mx^{m-1}(1 - x^k)} \geq x^k$$

and it is equivalent to

$$\frac{k(1 - x^m)}{m(1 - x^k)} \geq x^m \quad (14)$$

Therefore it is enough to show that

$$m(1 - x^k) < k(1 - x^m)$$

For $x = 1$, *both sides of the last inequality are the same. The derivative of the LHS is* $-kmx^{k-1}$ *and the derivative of the RHS is* $-kmx^{m-1}$. *Since* $-kmx^{k-1} > -kmx^{m-1}$ *the inequality (14) is satisfied.*

It is important to note that Proposition 7 provides a sufficient condition but not a necessary one for allocating the stronger agent in the first stage and the weaker agent in the second one. Allocating the stronger agent in the first stage most probably holds for a significantly larger group of distribution functions than the group determined by our sufficient condition.

5 Concluding remarks

We studied a two-stage sequential search with two agents who compete for one job. We assumed that the agents are ex-ante asymmetric, namely, their abilities are derived from asymmetric distribution functions. The designer who does not know the agent's abilities but only the distribution of their abilities imposes an ability threshold in every stage according to the agent who arrives in that stage. We demonstrate that the ratio between the ability thresholds of both stages depends first on the agents' distributions of types and second on the timing of play. Thus, while when agents are ex-ante symmetric the ability threshold levels are necessarily decreasing along the stages, in our model when the agents are ex-ante asymmetric they may increase. We also provided sufficient conditions such that the designer would prefer that the stronger agent will arrive first and the weaker one later. We believe that this result holds even for much weaker conditions than our sufficient conditions. In this paper, for simplicity, we considered a two-agent model but most of our results about the ratio of the ability thresholds can be easily generalized to a model with any number of agents. On the other hand, a generalization of our results about the optimal order of agents to sequential search models with any number of agents would most likely be quite complex to undertake.

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