FIRST-MOVER ADVANTAGE IN ROUND-ROBIN TOURNAMENTS

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First-Mover Advantage in Round-Robin Tournaments

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Abstract

We study round-robin tournaments with either three or four symmetric players whose values of winning are common knowledge. In the round-robin tournament with three players there are three stages, each of which includes one match between two players. The player who wins in two matches wins the tournament. We characterize the sub-game perfect equilibrium and show that each player maximizes his expected payoff and his probability to win if he competes in the first and the last stages of the tournament. In the round-robin tournament with four players there are three rounds, each of which includes two sequential matches where each player plays against a different opponent in every round. We characterize the sub-game perfect equilibrium and show that a player who plays in the first match of each of the first two rounds has a first-mover advantage as reflected by a significantly higher winning probability as well as a significantly higher expected payoff than his opponents.

JEL Classifications: D44, O31

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1 Introduction

Sequential all-pay (auctions) contests have been intensively studied as they have many real-life applications including political lobbying (Becker 1983), patent races (Wright 1983), R&D races (Dasgupta 1986), job promotion (Rosen 1986) and others. Most studies dealing with sequential all-pay contests assume a two-stage contest under complete information. Leininger (1991) modeled a patent race between an incumbent and an entrant as a sequential asymmetric all-pay contest under complete information, and Konrad and Leininger (2007) characterized the equilibrium of the all-pay contest under complete information in which a group of players choose their effort 'early' and the other group of players choose their effort 'late'. On the other hand, Segev and Sela (2014a,b,c) studied sequential all-pay contests under incomplete information in which every player plays only in one stage of the contest. They found a first-mover disadvantage in their model and suggested giving a head-start to the player in the first stage. We study here a more complicated form of multi-stage contests in which every player plays in each stage of the contest. This form of multi-stage contest is known as the round-robin tournament.

Sportive events are commonly organized as round-robin tournaments, two well known examples being professional football and basketball leagues. In the round-robin tournament, every individual player or team competes against all the others and in every stage a player plays a pair-wise match against a different opponent. Sometimes sportive events can also be organized as a combination of a round-robin tournament in the first part of the season and then as an elimination tournament in the second part where in the elimination tournament, players play pair-wise matches and the winner advances to the next round while the loser is eliminated from the competition. Examples of such combinations include US-Basketball, NCAA College Basketball, the FIFA (soccer) World Cup Playoffs and the UEFA Champions' League. The elimination tournament structure has been widely analyzed in the literature on contests. For example, Rosen (1986) studied an elimination tournament with homogeneous players where the probability of winning a match is a stochastic function of the players' efforts. Gradstein and Konrad (1999) and Harbaugh and Klumpp (2005) studied a rent-seeking contest à la Tullock (with homogenous players). Groh et al. (2012) studied an elimination tournament with four asymmetric players where players are matched in the all-pay auction in each of the stages and they found optimal seedings for different criteria. In contrast to elimination tournaments, the literature on round-robin tournaments seems to be quite sparse, the reason being the complexity of its analysis. This paper attempts to fill this gap by studying three-player and four-player round-robin tournaments with three stages where in each of the stages, a player competes against a different opponent in the all-pay auction.[1]

The outcomes of sequential contests such as round-robin tournaments are obviously affected by the timing of the play, namely, the order of the players in the contest. In other words, the allocation of players in the sequential contest affects their probabilities to win as well as their expected payoffs. In the round-robin tournaments we study here, we will show that the first mover has a meaningful advantage.

In the round-robin tournaments with three players every player competes against all the others and in every stage two players compete against each other in an all-pay contest. Thus, there are three rounds where in each round only one match takes place. We characterize the sub-game perfect equilibrium of the three-player round-robin tournament when the players are symmetric, namely, they have the same value of winning the tournament. We prove that the expected payoff of each player is maximized when he competes in the first and the last stages of the tournament. This result is not straightforward since it is not clear why a player prefers to play in the last stage while there is a positive probability that the winner of the tournament will be decided before the last (third) stage and then there is not any meaning to the match in that stage. However, the intuition for this result is that a player who wins the first match has an advantage and he prefers to play in the last stage since there is a significant probability that his opponent in the last stage will not have an incentive to compete and then he will win the tournament without wasting much effort.

In the round-robin tournaments with four players every player competes against all the others and in every stage a player plays a pair-wise match against a different opponent in an all-pay contest. There are three rounds where in each round two matches take place. The two matches in each round are scheduled one

[1] Three-player round-robin tournaments can be found in the real life, for example, the badminton tournament in the Olympic Games, London 2012, was organized in the form of a three-player round-robin tournament. In addition, three-player round-robin tournaments are also used in soccer, rugby and even in debates competitions. Four-player round-robin tournaments are very common in soccer, basketball, tennis and many other sport branches.

after another as we can see in many real-life round-robin tournaments. Thus, we have six different matches that take place one after another in three rounds such that in every round there are two sequential matches. In this case there is always one player who plays in the first match of the first two rounds. We show that this allocation allows the possibility of a significant first mover advantage.

We characterize the sub-game perfect equilibrium of the round-robin tournament with four symmetric players and prove that the player who plays in the first match in each of the first two rounds, namely, matches 1 and 3, has a significantly higher probability to win the tournament as well as a significantly higher expected payoff than his opponents. Although all the four players are ex-ante symmetric, the player who plays in the first match of each of the two rounds has a winning probability that is more than twice higher than the player with the second highest probability of winning and he also has an expected payoff that is more than seven (!) times higher than the player with the second highest expected payoff. Thus, we conclude that in round-robin tournaments with four players a contest designer should consider scheduling all the matches in the same round at the same time in order to obstruct any possible meaningful advantage to one of the players.

The intuition behind the above result is that if the first mover in the first two rounds wins, the rest of the players will be discouraged since even if they win in the first matches their probabilities of winning as well as their expected payoff will be lower than that of the first mover. This creates ahead-behind asymmetry, which decreases players' efforts and therefore increases the asymmetry of winning probabilities and expected payoffs between the player with the first mover advantage and the other players.

Another interesting result that we find is that the order of matches in the last round has no effect on the players' winning probabilities and their expected payoffs. This result is quite surprising given that although in our round-robin tournaments there are only three rounds, independent of the results of the first two rounds, the matches in the final round do not really affect the final results. The intuition for this result derives from the previous one about the first-mover advantage. Since the first-mover advantage is so strong the tournament is (almost) decided with a relatively high probability before the last round such that the matches in the last round do not affect the final outcome. Thus, against our intuition, we conclude that if there are attractive matches between two opponents, it is better to allocate them in one of the first two rounds of the round-robin tournament.

The existence of the first mover advantage has sparked much heated debate in both the theoretical and empirical economic literature. According to the theoretical studies of Kingston (1976) and Anderson (1977), in a contest between two players, a player who has the first mover advantage in the best of k ($k \geq 3$ is an odd number) stages has a higher probability to win than his opponent no matter how the moves are alternated.[2] Likewise a field study performed by Magnus and Klaasen (1999) revealed that serving first in the first set in the Wimbledon Grand Slam tennis tournament provides an advantage to win the set. And in another paper, Apestigua and Palacios-Huerta (2010) found that in soccer penalty shoot-outs, the first-kicking team has a significant margin of twenty one percent points over the second team. However, on a different sample of soccer shoot-outs Kocher, Lenz and Sutter (2012) found that the first-kicking team's winning percentage was not significantly different from fifty percent. In addition, in an experimental study involving young Italian basketball players, Feri, Innocenti and Pin (2013) found no first mover advantage in a two-player free-throw shooting contest in which the leader shoots five baskets one after another and then the follower shoots his five baskets. Moreover, they observed that second movers performed significantly better under psychological pressure. This same second mover advantage was found in an empirical study of Page and Page (2007) who showed that there is advantage of playing in the second home leg game in soccer European Cups tournaments. Krumer (2013) explained their result theoretically by assuming existence of a psychological advantage.

Our paper is also related to the statistical literature on the design of various forms of tournaments. The pioneering paper[3] is David (1959) who considered the winning probability of the top player in a four player tournament with a random seeding. This literature assumes that, for each match among players i and j, there is a fixed, exogenously given probability that i beats j. In particular, this probability does not depend on the stage of the tournament where the particular match takes place nor on the identity of the expected opponent at the next stage. In contrast, in our round-robin model each match among two players is an all-pay auction. As a result, winning probabilities in each match become endogenous in that they

[2] The analysis of our model is related to the analysis of the best-of-k tournaments (see, Konrad and Kovenock (2009), Malueg and Yates (2010), Sela (2011) and Krumer (2013)) in which the winner is the one who is first to win $\frac{k+1}{2}$ games.

[3] See also Glenn (1960) and Searles (1963) for early contributions.

result from mixed equilibrium strategies, and are positively correlated to win valuations. Moreover, the win probabilities depend on the stage of the tournament where the match takes place, and on the identity of the future expected opponents.

The paper is organized as follows: Section 2 presents the equilibrium analysis of the round-robin tournament with three symmetric players and Section 3 presents the equilibrium analysis of the round-robin tournament with four symmetric players. Section 4 concludes. All the possible paths in the tournaments are presented by tree games in Appendix A and some of the calculations appear in Appendix B.

2 The round-robin tournament with three players

Consider three symmetric players (or teams) $i = 1, 2, 3$ who compete in a round-robin all-pay tournament. In each stage t, $t = 1, 2, 3$ there is a different pair-wise match such that each player competes in two different stages. The player who wins two matches wins the tournament and in the case that each player wins only once, each of them wins the tournament with the same probability. If one of the players wins in the first two stages, the winner of the tournament is then decided and the players in the last stage exert efforts that approach zero. We model each match among two players as an all-pay auction; both players exert efforts, and the one exerting the higher effort wins. Without loss of generality assume that player i's value of winning the tournament is $v = 1$ and his cost function is $c(x_i) = x_i$, where x_i is his effort.

We first explain how players' strategies are calculated in each match of the tournament. Suppose that players i and j compete in match $g, g = 1, 2, 3$. We denote by p_{ij} the probability that player i wins the match against player j and by E_i, E_j the expected payoffs of players i and j, respectively. The mixed strategies of the players in game g will be denoted by $F_{kg}(x), k = i, j$. Assume now that if player i wins in this match, his conditional expected payoff in the tournament is w_{ig} given the previous outcomes and the possible future outcomes. Similarly, if player i loses in this match, his conditional expected payoff in the tournament is l_{ig}. Without loss of generality, assume that $w_{ig} - l_{ig} > w_{jg} - l_{jg}$. Then, according to Baye, Kovenock and de Vries (1996), there is always a unique mixed-strategy equilibrium in which players i and j randomize on the interval $[0, w_{jg} - l_{jg}]$ according to their effort cumulative distribution functions, which are given by

$$\begin{aligned} E_i &= w_{ig}F_{jg}(x) + l_{ig}(1 - F_{jg}(x)) - x = l_{jg} + w_{ig} - w_{jg} \\ E_j &= w_{jg}F_{ig}(x) + l_{jg}(1 - F_{ig}(x)) - x = l_{jg} \end{aligned}$$

Thus, player i's equilibrium effort in match g is uniformly distributed; that is

$$F_{ig}(x) = \frac{x}{w_{jg} - l_{jg}}$$

while player j's equilibrium effort is distributed according to the cumulative distribution function

$$F_{jg}(x) = \frac{l_{jg} - l_{ig} + w_{ig} - w_{jg} + x}{w_{ig} - l_{ig}}$$

Player $j's$ probability to win against player i is then

$$p_{ji} = \frac{w_{jg} - l_{jg}}{2(w_{ig} - l_{ig})}$$

In order to analyze the sub-game perfect equilibrium of the round-robin tournament with three symmetric players we begin with the last stage of the tournament and go backwards to the previous stages. Figure 1 presents the symmetric round-robin tournament as a tree game. We denote by p_{ij}^{*} the probability that player i wins against player j in vertex $*$ of the tree game. [Figure 1 about here].

2.1 Stage 3 - player 2 vs. player 3

Players 2 and 3 compete in the last stage only if at least one of them won in the previous stages. Thus, we have the following three scenarios:

1. Assume first that player 2 won the match in the first stage and player 3 won the match in the second stage (vertex A in Figure 1). Then if each of the players wins in stage 3, he also wins the tournament. Thus, following Hillman and Riley (1989) and Baye, Kovenock and de Vries (1996), there is always a unique mixed strategy equilibrium in which both players randomize on the interval $[0, 1]$ according to their cumulative distribution functions $F_i^{(3)}, i = 2, 3$ which are given by

$$1 \cdot F_i^{(3)}(x) - x = 0 \quad i = 2, 3 \tag{1}$$

Then, player 2's probability to win in the third stage is

$$p_{23}^{A} = 0.5$$

2. Assume now that player 2 won the match in the first stage and player 3 lost the match in the second stage (vertex B in Figure 1). Then, if player 2 wins in this stage, he wins the tournament and his payoff is 1, whereas player 3's payoff is zero. But, if player 3 wins in this stage, then each of the players has exactly one win, and then each of the players has an expected payoff of 1/3. Thus, we obtain that players 2 and 3 randomize on the interval $[0, 1/3]$ according to their effort cumulative distribution functions $F_i^{(3)}, i = 2, 3$ which are given by

$$\begin{aligned} 1 \cdot F_3^{(3)}(x) + \frac{1}{3} \cdot (1 - F_3^{(3)}(x)) - x &= \frac{2}{3} \\ \frac{1}{3} \cdot F_2^{(3)}(x) - x &= 0 \end{aligned} \tag{2}$$

Then, player 2's probability to win in the third stage is

$$p_{23}^{B} = 1 - \frac{1}{4} = 0.75$$

3. Finally, assume that player 2 lost the match in the first stage and player 3 won the match in the second stage (vertex C in Figure 1). Then, similarly to the previous case, we obtain that players 2 and 3 randomize on the interval $[0, 1/3]$ according to their effort cumulative distribution functions $F_i^{(3)}, i = 2, 3$ which are now given by

$$\begin{aligned} 1 \cdot F_2^{(3)}(x) + \frac{1}{3} \cdot (1 - F_2^{(3)}(x)) - x &= \frac{2}{3} \\ \frac{1}{3} \cdot F_3^{(3)}(x) - x &= 0 \end{aligned} \tag{3}$$

Then, player $2'$s probability to win in the third stage is

$$p_{23}^{C} = 0.25$$

2.2 Stage 2 - player 1 vs. player 3

Based on the results of the match in the first stage, we have two possible scenarios:

1. Assume first that player 1 lost the match in the first stage (vertex D in Figure 1). Then, if player 3 wins in this stage, by (1) his expected payoff in the next stage is zero. If player 3 loses in this stage, by (2) his expected payoff is zero as well. Thus, in such a case, player 3 has no incentive to exert a positive effort and player 1 wins in this stage with a probability of one.[4]

2. Assume now that player 1 won the match in the first stage (vertex E in Figure 1). Then, if he wins again in this stage he also wins the tournament and therefore his payoff is 1. The other players' payoffs are then zero. However, if player 1 loses in this stage, then by (3) player 3's expected payoff is 2/3 and player 1's expected payoff depends on the result of the match between players 2 and 3 in the last stage. If player 3 wins in the last stage, which happens with a probability of 0.75, player 1's expected payoff is zero. On the other hand, if player 2 wins in the last stage which happens with a probability of 0.25, each of the players has one win and therefore an expected payoff of 1/3. In sum, if player 1 loses in this stage, his expected payoff is 1/12.

Thus, we obtain that players 1 and 3 randomize on the interval $[0, 2/3]$ according to their effort cumulative distribution functions $F_i^{(2)}, i = 1, 3$ which are given by

$$\begin{aligned} 1 \cdot F_3^{(2)}(x) + \frac{1}{12} \cdot (1 - F_3^{(2)}(x)) - x &= \frac{1}{3} \\ \frac{2}{3} \cdot F_1^{(2)}(x) - x &= 0 \end{aligned} \tag{4}$$

Then, player 1's probability to win in the second stage is

$$p_{13}^E = 1 - \frac{8}{22} = \frac{7}{11}$$

2.3 Stage 1 - player 1 vs. player 2

If player 1 wins the match in the first stage (vertex F in Figure 1), by (4) his expected payoff in the next stage is 1/3. But if player 1 loses the match in the first stage, he has an expected payoff of 1/3 only if he wins in the second stage which happens with a probability of one, and player 2 loses against player 3 in the

[4] It is important to note that when a player has no incentive to exert a positive effort we actually do not have an equilibrium. However, in order to solve this problem, similarly to Groh et al. (2012), we can assume that each player obtains a payment $k > 0$, independent from his performance, and then we consider the limit behavior as $k \to 0$. This assumption does not affect the players' behavior in our model but ensures the equilibrium existence.

last stage which happens with a probability of 0.25. Thus, if player 1 loses in the first stage his expected payoff in the next stage is 1/12.

Now, if player 2 wins the match in the first stage (vertex F in Figure 1), player 1 wins for sure in the second stage and then by (2) player 2's expected payoff is 2/3. However, if player 2 loses the match in the first stage, and player 1 wins also in the second stage player 2 has an expected payoff of zero. Furthermore, even if player 1 loses in the second stage, by (3) player 2 has an expected payoff of zero. Thus, we obtain that players 1 and 2 randomize on the interval $[0, 1/4]$ according to their effort cumulative distribution functions $F_i^{(1)}, i = 1, 2$ which are given by

$$\begin{aligned} \frac{1}{3} \cdot F_2^{(1)}(x) + \frac{1}{12} \cdot (1 - F_2^{(1)}(x)) - x &= \frac{1}{12} \\ \frac{2}{3} \cdot F_1^{(1)}(x) - x &= \frac{5}{12} \end{aligned} \tag{5}$$

Then, player 1's probability to win in the first stage is

$$p_{12}^F = \frac{3}{16}$$

By the above analysis we obtain:

Proposition 1 *In the sub-game perfect equilibrium of the round-robin tournament with three symmetric players, the players' expected payoffs are as follows: player 1's expected payoff is* 1/12, *player 2's is* 5/12, *and player 3's is zero.*

By the above analysis we also obtain:

Proposition 2 *In the sub-game perfect equilibrium of the round-robin tournament with three symmetric players, the players' probabilities to win the tournament are as follows:*

Player 1's probability to win is

$$P_1 = p_{12}^F \cdot p_{13}^E + \frac{p_{12}^F \cdot p_{31}^E \cdot p_{23}^C}{3} + \frac{p_{21}^F \cdot p_{13}^D \cdot p_{32}^B}{3} = 0.193$$

Player 2's probability to win is

$$P_2 = p_{21}^F \cdot p_{13}^D \cdot p_{23}^B + p_{21}^F \cdot p_{31}^D \cdot p_{23}^A + \frac{p_{12}^F \cdot p_{31}^E \cdot p_{23}^C}{3} + \frac{p_{21}^F \cdot p_{13}^D \cdot p_{32}^B}{3} = 0.682$$

and player 3's probability to win is

$$P_3 = p_{12}^F \cdot p_{31}^E \cdot p_{32}^C + p_{21}^F \cdot p_{31}^D \cdot p_{32}^A + \frac{p_{12}^F \cdot p_{31}^E \cdot p_{23}^C}{3} + \frac{p_{21}^F \cdot p_{13}^D \cdot p_{32}^B}{3} = 0.125$$

By Propositions 1 and 2 we can conclude that

Theorem 1 *In the round-robin tournament with three symmetric players, the player who competes in the first and the last stages has the highest probability to win the tournament as well as the highest expected payoff.*

Theorem 1 demonstrates the first-mover advantage in the round-robin tournament with three symmetric players where the player who does not play in the first stage (player 3) has the lowest probability to win the tournament as well as the lowest expected payoff. In the next section we show that the first-mover advantage is even stronger in the round-robin tournament with four symmetric players.

3 The round-robin tournament with four players

Consider four symmetric players (or teams) competing for a single prize in the round-robin tournament. Without loss of generality assume that the players' value of winning the tournament is $v = 1$ and this value is commonly known. As previously, the players play pair-wise matches and each match between two players is modelled by an all-pay contest where both players simultaneously exert efforts, and the player with the higher effort wins the match. In this tournament the players compete one time against each of their opponents in sequential matches, such that every player plays three matches. We consider three rounds, denoted by $r = 1, 2, 3$, where each player plays one match in each round, and there are two sequential matches in each round. Thus, there are six different matches in the tournament denoted by $g = 1, 2, 3, 4, 5, 6$. Player i's cost in match g is $c(x_{ig}) = x_{ig}$ where x_{ig} is his effort. A player that wins the highest number of matches wins the tournament. In the case that two or more players have the same highest number of wins, there will be a draw to determine the winner of the tournament. If one of the players has three wins before the last match, the winner of the tournament is decided and the players exert efforts that approach zero in the later matches.

Suppose that players i and j compete in match $g, g = 1, 2, 3, 4, 5, 6$. As in the previous section we denote by p_{ij} the probability that player i wins the match against player j and by E_i, E_j the expected payoffs of players i and j, respectively. The mixed strategies of the players in game g will be denoted by $F_{kg}(x), k = i, j$.

While in a round robin tournament with four asymmetric players there are many possible allocations of players in the six matches, in our model with four symmetric players there are only two different allocations of players. The first possible allocation is when one of the players always plays in the first match of each round, namely, he plays in matches 1, 3 and 5. In the second, one of the players always plays in the second match of each round, namely, he plays in matches 2,4 and 6. Any other allocation of the players is equivalent to one of these two possible allocations because of the symmetry among the players. Below we analyze the sub-game equilibrium in the round robin tournament with four players for each possible allocation of players. In each possible allocation we calculate for every possible match the players' strategies, their expected payoffs and their probabilities of winning.

3.1 Case A: One of the players always plays in the first match of each round

Assume that player 1 always plays in the first match of each round. Then, without loss of generality, the order of the games is

Round 1:	Game 1: player 1 - player 2
	Game 2: player 3 - player 4
Round 2:	Game 3: player 1 - player 3
	Game 4: player 2 - player 4
Round 3:	Game 5: player 1 - player 4
	Game 6: player 2 - player 3

Figures 2 and 3 in Appendix A present all the possible paths of this tournament. [Figures 2 and 3 about here]

As in the previous section in order to analyze the sub-game perfect equilibrium of the round-robin tournament with four players we begin with the last match of the tournament and go backwards to the previous matches. Because of the complexity of the analytical analysis, we provide only the final results of this analysis in Table 1 (Appendix B). These include the players' mixed strategies, their expected payoffs as well as their winning probabilities in each vertex (match) of the tree game given by Figures 2 and 3 (Appendix A). Similarly to the previous section we can assume that each player obtains a payment of $k > 0$ when he wins a single match, and then we can consider the limit behavior as $k \to 0$. This assumption does not affect the players' behavior in our model, but rather serves to ensure the existence of equilibrium. The first result provides the ranking of the players' winning probabilities and their expected payoffs and emphasizes the first mover advantage.

Proposition 3 *In the sub-game perfect equilibrium of the round-robin tournament with four symmetric players, if player 1 plays in the first match of each of the rounds he has the highest expected payoff as well as the highest probability to win the tournament.*

Proof. By the analysis given in Table 1 (Appendix B) of the sub-game perfect equilibrium of the round-robin tournament with four symmetric players when player 1 always plays in the first match of each round (games 1, 3 and 5), player 2 plays in matches 1,4 and 6, player 3 plays in matches 2, 3 and 6, and player 4 plays in matches 2, 4 and 5, the players' expected payoffs and their winning probabilities are

Player	Expected payoff	Winning probability
1	0.3	0.621
2	0.039	0.051
3	0.009	0.252
4	0.001	0.076

■

The intuition behind Proposition 3 can be explained by the first mover advantage. If player 1 wins in the first match of the first round, then if his next opponent in the next round (player 3) also wins in the first round, they both have the same probability to win in the second round. However, if his next opponent loses in the first round then his probability to win against player 1 is extremely low. Moreover, if player 1 wins in the first match of the first round, then there is no chance that the winner of the tournament will be decided before player 1's last match (game 5). On the other hand, if player 2 wins against player 1 in the first match of the first round, there is still a positive probability that the winner of the tournament will be decided before the last match of player 2 (Vertexes 38 and 40 in Figures 2 and 3). Furthermore, even if player 2 wins in the first match against player 1, player 3, and not player 2, will have the first mover advantage since he will play in the first match of round 2 (game 3) against player 1, who has already lost one match and now he becomes the underdog in the next match against player 3. Therefore, even if player 2 wins in the first match against player 1, it doesn't make him the favorite. This situation discourages player 2 and reduces his (costly) exerted efforts in the first match such that player 1 wins with relatively high probability in the first round which gives him an advantage over his opponents also in the following rounds.

3.2 Case B: One of the players always plays in the last match of each round

Assume now that player 4 always plays in the second match of each round. Then, without loss of generality, the order of the games is

Round 1:	Game 1: player 1 - player 2
	Game 2: player 3 - player 4
Round 2:	Game 3: player 1 - player 3
	Game 4: player 2 - player 4
Round 3:	Game 5: player 2 - player 3
	Game 6: player 1 - player 4

Figures 4 and 5 (Appendix A) present all the possible paths of this tournament, and Table 2 (Appendix B) provides the calculations of the players' expected payoffs and their winning probabilities. [Figures 4 and 5 about here]; [Table 2 about here]. A comparison of the results given by Tables 1 and 2 reveals that the players' expected payoffs and their probabilities of winning in Case B are the same as in Case A. Therefore we obtain the following main result.

Theorem 2 *In the sub-game perfect equilibrium of the round-robin all-pay tournament with four symmetric players, the player who plays in the first matches of each of the first two rounds has the highest expected payoff as well as the highest probability to win the tournament.*

It is important to emphasize that according to Theorem 2 the player who plays in the first matches of the first two rounds has a winning probability that is 2.5 (!) times higher than the player with the second highest probability of winning and an expected payoff that is 7.7 (!) times higher than the player with the second highest expected payoff. Hence, the first-mover advantage in the round-robin tournament with four symmetric players is quite dramatic and affects the players' ex-ante probabilities to win the tournament.

If we compare the order of the games in cases A and B we can see that the difference between them is only in the last round. Thus, given that the players' expected payoffs and their probabilities of winning in Case B are the same as in Case A we obtain the following result.

Proposition 4 *In the sub-game perfect equilibrium of the round-robin tournament with four symmetric players, the order of the games in the last round of the tournament (games 5 and 6) has no effect on the players' expected payoffs as well as on their winning probabilities.*

The intuition behind Proposition 4 is that sequential contests are sometimes decided before the last stage or they are almost surely decided such that the games in the last stages are completely not equal. Indeed, in Case A there are 7 sub-cases in which the winner of the tournament is decided before the last match (Vertexes 25, 26, 29, 30, 32, 38 and 40 in Figures 2 and 3). Moreover, in 12 other sub-cases the last match (game 6) occurs with a probability of zero (Vertexes 1, 4, 8, 9, 12, 13, 14, 15, 16, 17, 22, 24 in Figures 2 and 3) and in 4 other sub-cases, even the first match in the last round (game 5) occurs with a probability of zero (Vertexes 30, 31, 34 and 35 in Figures 2 and 3).

In Case B, we have a similar situation. There are 7 sub-cases in which the winner of the tournament is decided before the last match (Vertexes 27, 28, 33, 35, 36, 37, 39 in Figures 4 and 5), 12 other sub-cases in which the last game occurs with a probability of zero (Vertexes 3, 6, 8, 9, 10, 11, 12, 13, 16, 17, 19 and 21 of Figures 4 and 5) and 4 other sub-cases in which the first match of the last round (game 5) occurs with a probability of zero (Vertexes 30, 31, 34 and 35 of Figures 4 and 5).

4 Concluding remarks

We first analyzed the sub-game perfect equilibrium of the round-robin tournaments with three asymmetric players. We showed that a player's expected payoff is maximized when he plays in the first and the last stages. We then analyzed the sub-game perfect equilibrium of the round-robin tournament with four symmetric players. We showed that a player who plays in the first match of each of the first two rounds has a significantly higher probability to win as well as a significantly higher expected payoff than his opponents. These results emphasize the first mover advantage in the round-robin tournaments and thus raises the question of fairness in this form of tournaments. Therefore, even though the contest designer wishes to increase his revenue, in light of the fair play principle, in the round-robin tournament with four players he might want to allocate all the matches in the same round at the same time. The analysis of such a round-robin tournament where all the matches at the same round are at the same time is much more complicated than the analysis of our model where the matches are sequential in each round. Thus, a comparison of these two structures of round-robin tournaments with four players is not simple.

We also found that the order of the matches in the last round of the tournament with four players has no effect on players' winning probabilities and their expected payoffs. The reason is that there is a high probability that the tournament will be decided before the last round and then some of the players will have no real incentive to compete in the last round.

Further research could be extended to include several prizes in order to investigate whether the first mover advantage exists in multi-prize round-robin tournaments. It would also be of interest to examine our results in a laboratory setting or using real-world data.

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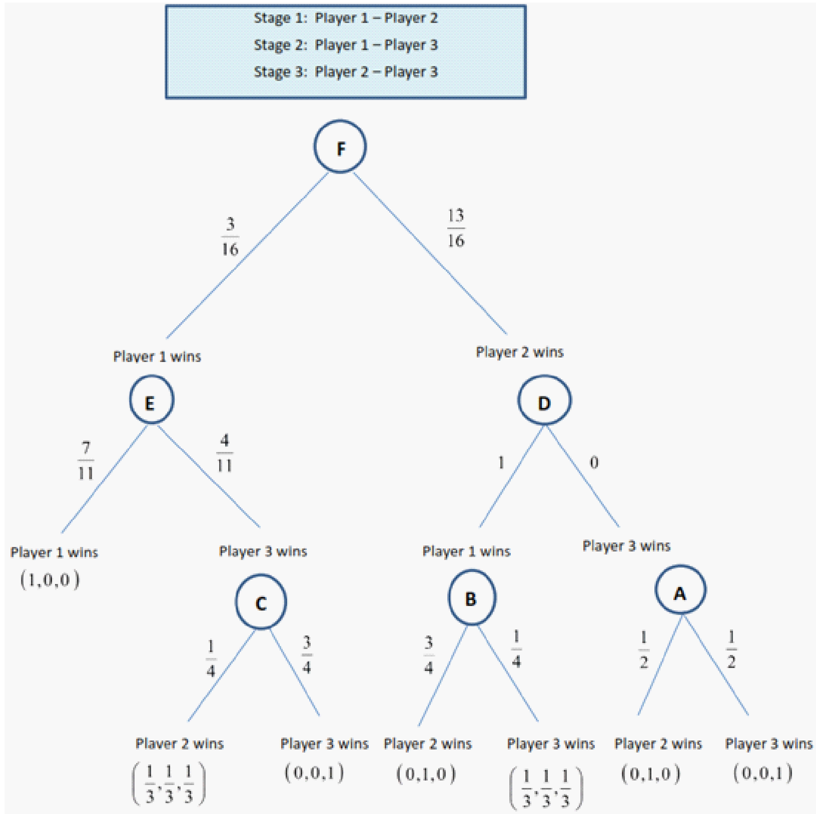

Figure 1: The tree game of the round-robin tournament with three symmetric players.

5 Appendix A

We present in Figure 1 all the possible paths of the round-robin tournament with three symmetric players, and in Figures 2,3,4 and 5 we present the tree game of the round-robin tournaments for the two possible allocations of players in the round-robin tournament with four symmetric players (Case A and Case B). Each tree game describes all the possible paths in the round-robin tournament. Since there are 55 possible matches (vertexes) in the tournament with four players, each tree game is exceedingly large and we have to divide it into two parts.

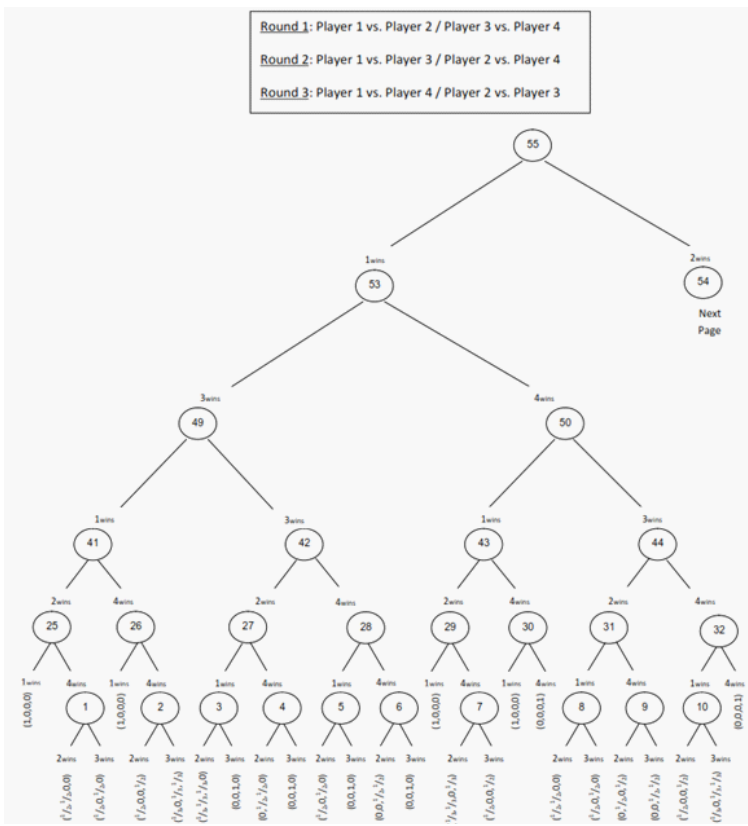

Figure 2: Part I of the tree game in Case A of the round-robin tournament with four symmetric players.

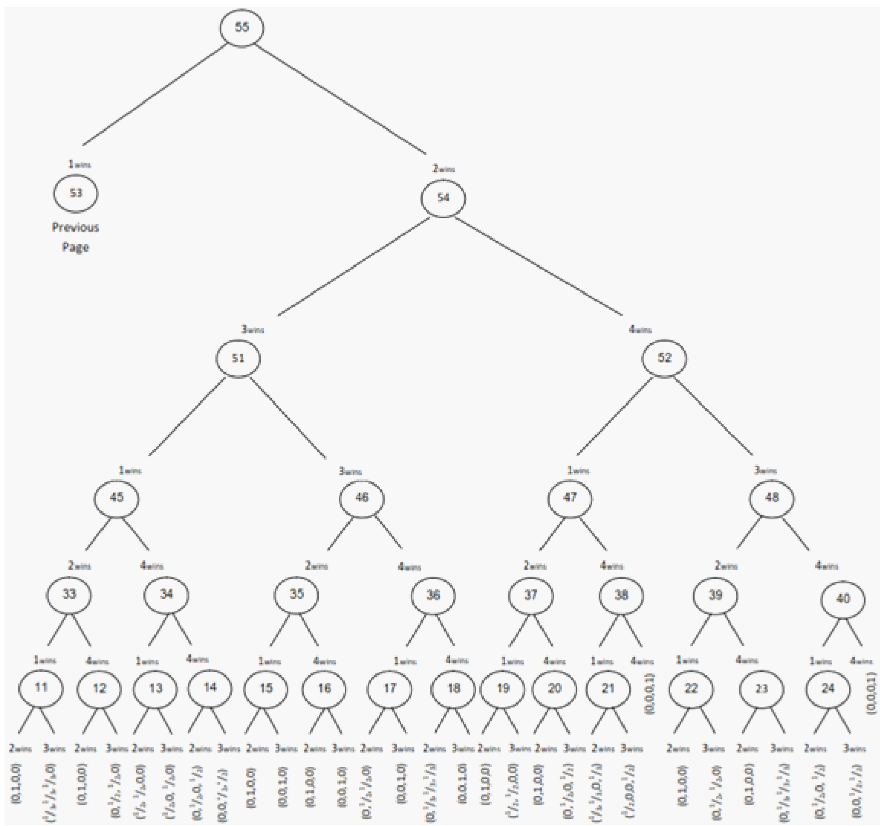

Figure 3: Part II of the tree game in Case A of the round-robin tournament with four symmetric players.

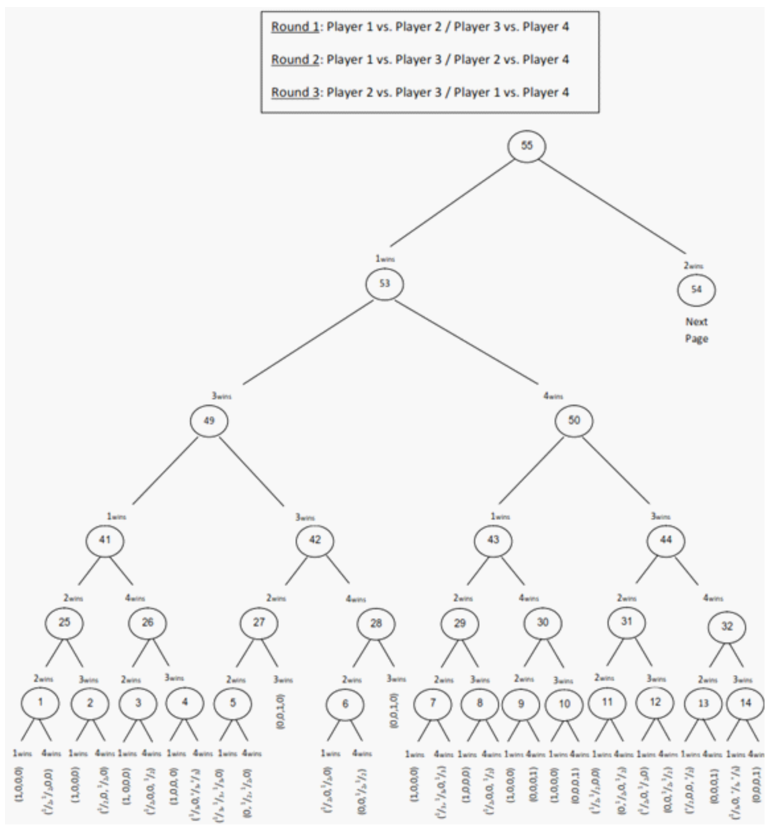

Figure 4: Part I of the tree game in Case B of the round-robin tournament with four symmetric players.

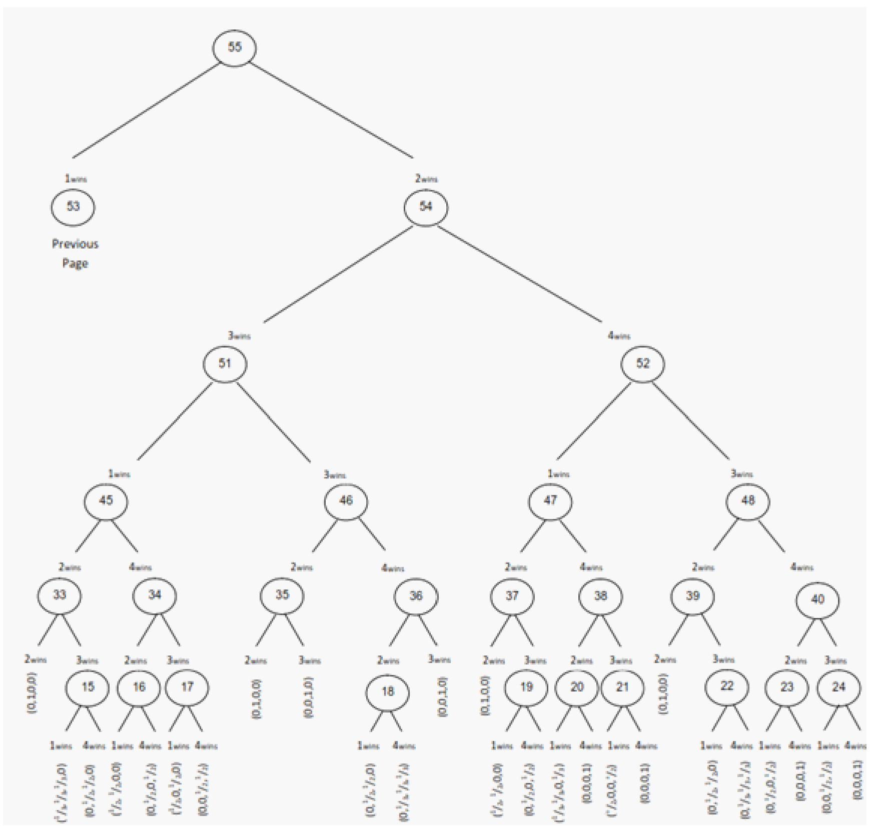

Figure 5: Part II of the tree game in Case B of the round-robin tournament with four symmetric players.

6 Appendix B

In the following, we provide in every possible vertex (match) the players' mixed-strategies, their expected payoffs and their probabilities of winning. These results are summarized in Table 1 (Case A) and Table 2 (Case B) each of which includes 55 vertexes. We provide first the expected payoffs and winning probabilities of Case A by Table 1 and then of Case B in Table 2.

Vertex 1: $E_2 = \frac{1}{2} \cdot F_{36}(x) - x = 0$ $E_3 = \frac{1}{2} \cdot F_{26}(x) - x = 0$ $p_{23} = \frac{1}{2}$	Vertex 2: $E_2 = 0 \cdot F_{36}(x) - x = 0$ $E_3 = \frac{1}{3} \cdot F_{26}(x) - x = \frac{1}{3}$ $p_{23} = 0$
Vertex 3: $E_2 = \frac{1}{3} \cdot F_{36}(x) - x = 0$ $E_3 = 1 \cdot F_{26}(x) + \frac{1}{3} \cdot (1 - F_{26}(x)) - x = \frac{2}{3}$ $p_{23} = \frac{1}{4}$	Vertex 4: $E_2 = \frac{1}{2} \cdot F_{36}(x) - x = 0$ $E_3 = 1 \cdot F_{26}(x) + \frac{1}{2} \cdot (1 - F_{26}(x)) - x = \frac{1}{2}$ $p_{23} = \frac{1}{2}$
Vertex 5: $E_2 = 0 \cdot F_{36}(x) - x = 0$ $E_3 = 1 \cdot F_{26}(x) + \frac{1}{2} \cdot (1 - F_{26}(x)) - x = 1$ $p_{23} = 0$	Vertex 6: $E_2 = 0 \cdot F_{36}(x) - x = 0$ $E_3 = 1 \cdot F_{26}(x) + \frac{1}{2} \cdot (1 - F_{26}(x)) - x = 1$ $p_{23} = 0$
Vertex 7: $E_2 = \frac{1}{3} \cdot F_{36}(x) - x = \frac{1}{3}$ $E_3 = 0 \cdot F_{26}(x) - x = 0$ $p_{23} = 1$	Vertex 8: $E_2 = \frac{1}{2} \cdot F_{36}(x) - x = 0$ $E_3 = \frac{1}{2} \cdot F_{26}(x) - x = 0$ $p_{23} = \frac{1}{2}$
Vertex 9: $E_2 = \frac{1}{2} \cdot F_{36}(x) - x = 0$ $E_3 = \frac{1}{2} \cdot F_{26}(x) - x = 0$ $p_{23} = \frac{1}{2}$	Vertex 10: $E_2 = 0 \cdot F_{36}(x) - x = 0$ $E_3 = \frac{1}{3} \cdot F_{26}(x) - x = \frac{1}{3}$ $p_{23} = 0$
Vertex 11: $E_2 = 1 \cdot F_{36}(x) + \frac{1}{3} \cdot (1 - F_{36}(x)) - x = \frac{2}{3}$ $E_3 = \frac{1}{3} \cdot F_{26}(x) - x = 0$ $p_{23} = \frac{3}{4}$	Vertex 12: $E_2 = 1 \cdot F_{36}(x) + \frac{1}{2} \cdot (1 - F_{36}(x)) - x = \frac{1}{2}$ $E_3 = \frac{1}{2} \cdot F_{26}(x) - x = 0$ $p_{23} = \frac{1}{2}$

Table 1 : Players$'$ expected payoffs and winning probabilities in Vertexes 1 – 12 of Figures 2 – 3.

Vertex 13: $E_2 = \frac{1}{2} \cdot F_{36}(x) - x = 0$ $E_3 = \frac{1}{2} \cdot F_{26}(x) - x = 0$ $p_{23} = \frac{1}{2}$	Vertex 14: $E_2 = \frac{1}{2} \cdot F_{36}(x) - x = 0$ $E_3 = \frac{1}{2} \cdot F_{26}(x) - x = 0$ $p_{23} = \frac{1}{2}$
Vertex 15: $E_2 = 1 \cdot F_{36}(x) - x = 0$ $E_3 = 1 \cdot F_{26}(x) - x = 0$ $p_{23} = \frac{1}{2}$	Vertex 16: $E_2 = 1 \cdot F_{36}(x) - x = 0$ $E_3 = 1 \cdot F_{26}(x) - x = 0$ $p_{23} = \frac{1}{2}$
Vertex 17: $E_2 = \frac{1}{2} \cdot F_{36}(x) - x = 0$ $E_3 = 1 \cdot F_{26}(x) + \frac{1}{2} \cdot (1 - F_{26}(x)) - x = \frac{1}{2}$ $p_{23} = \frac{1}{2}$	Vertex 18: $E_2 = \frac{1}{3} \cdot F_{36}(x) - x = 0$ $E_3 = 1 \cdot F_{26}(x) + \frac{1}{3} \cdot (1 - F_{26}(x)) - x = \frac{2}{3}$ $p_{23} = \frac{1}{4}$
Vertex 19: $E_2 = 1 \cdot F_{36}(x) + \frac{1}{2} \cdot (1 - F_{36}(x)) - x = 1$ $E_3 = 0 \cdot F_{26}(x) - x = 0$ $p_{23} = 1$	Vertex 20: $E_2 = 1 \cdot F_{36}(x) + \frac{1}{2} \cdot (1 - F_{36}(x)) - x = 1$ $E_3 = 0 \cdot F_{26}(x) - x = 0$ $p_{23} = 1$
Vertex 21: $E_2 = \frac{1}{3} \cdot F_{36}(x) - x = \frac{1}{3}$ $E_3 = 0 \cdot F_{26}(x) - x = 0$ $p_{23} = 1$	Vertex 22: $E_2 = 1 \cdot F_{36}(x) + \frac{1}{2} \cdot (1 - F_{36}(x)) - x = \frac{1}{2}$ $E_3 = \frac{1}{2} \cdot F_{26}(x) - x = 0$ $p_{23} = \frac{1}{2}$
Vertex 23: $E_2 = 1 \cdot F_{36}(x) + \frac{1}{3} \cdot (1 - F_{36}(x)) - x = \frac{2}{3}$ $E_3 = \frac{1}{3} \cdot F_{26}(x) - x = 0$ $p_{23} = \frac{3}{4}$	Vertex 24: $E_2 = \frac{1}{2} \cdot F_{36}(x) - x = 0$ $E_3 = \frac{1}{2} \cdot F_{26}(x) - x = 0$ $p_{23} = \frac{1}{2}$

Table 1 : Players′ expected payoffs and winning probabilities in Vertexes 13 – 24 of Figure 3.

Vertex 25: $E_1 = 1 \cdot F_{45}(x) + \frac{1}{2} \cdot (1 - F_{45}(x)) - x = 1$ $E_4 = 0 \cdot F_{15}(x) - x = 0$ $p_{14} = 1$	Vertex 26: $E_1 = 1 \cdot F_{45}(x) + \frac{1}{3} \cdot (1 - F_{45}(x)) - x = \frac{2}{3}$ $E_4 = \frac{1}{3} \cdot F_{15}(x) - x = 0$ $p_{14} = \frac{3}{4}$
Vertex 27: $E_1 = \frac{1}{12} \cdot F_{45}(x) - x = \frac{1}{12}$ $E_4 = 0 \cdot F_{15}(x) - x = 0$ $p_{14} = 1$	Vertex 28: $E_1 = 0 \cdot F_{45}(x) - x = 0$ $E_4 = 0 \cdot F_{15}(x) - x = 0$ $p_{14} = \frac{1}{2}$
Vertex 29: $E_1 = 1 \cdot F_{45}(x) + \frac{1}{3} \cdot (1 - F_{45}(x)) - x = \frac{2}{3}$ $E_4 = \frac{1}{3} \cdot F_{15}(x) - x = 0$ $p_{14} = \frac{3}{4}$	Vertex 30: $E_1 = 1 \cdot F_{45}(x) - x = 0$ $E_4 = 1 \cdot F_{15}(x) - x = 0$ $p_{14} = \frac{1}{2}$
Vertex 31: $E_1 = \frac{1}{2} \cdot F_{45}(x) - x = 0$ $E_4 = \frac{1}{2} \cdot F_{15}(x) - x = 0$ $p_{14} = \frac{1}{2}$	Vertex 32: $E_1 = \frac{1}{3} \cdot F_{45}(x) - x = 0$ $E_4 = 1 \cdot F_{15}(x) + \frac{1}{3} \cdot (1 - F_{15}(x)) - x = \frac{2}{3}$ $p_{14} = \frac{1}{4}$
Vertex 33: $E_1 = \frac{1}{12} \cdot F_{45}(x) - x = \frac{1}{12}$ $E_4 = 0 \cdot F_{15}(x) - x = 0$ $p_{14} = 1$	Vertex 34: $E_1 = \frac{1}{2} \cdot F_{45}(x) - x = 0$ $E_4 = \frac{1}{2} \cdot F_{15}(x) - x = 0$ $p_{14} = \frac{1}{2}$
Vertex 35: $E_1 = 0 \cdot F_{45}(x) - x = 0$ $E_4 = 0 \cdot F_{15}(x) - x = 0$ $p_{14} = \frac{1}{2}$	Vertex 36: $E_1 = 0 \cdot F_{45}(x) - x = 0$ $E_4 = \frac{1}{12} \cdot F_{15}(x) - x = \frac{1}{12}$ $p_{14} = 0$

Table 1 : Players$'$ expected payoffs and winning probabilities in Vertexes 25 – 36 of Figures 2 – 3.

Vertex 37: $E_1 = 0 \cdot F_{45}(x) - x = 0$ $E_4 = 0 \cdot F_{15}(x) - x = 0$ $p_{14} = \frac{1}{2}$	Vertex 38: $E_1 = \frac{1}{3} \cdot F_{45}(x) - x = 0$ $E_4 = 1 \cdot F_{15}(x) + \frac{1}{3} \cdot (1 - F_{15}(x)) - x = \frac{2}{3}$ $p_{14} = \frac{1}{4}$
Vertex 39: $E_1 = 0 \cdot F_{45}(x) - x = 0$ $E_4 = \frac{1}{12} \cdot F_{15}(x) - x = \frac{1}{12}$ $p_{14} = 0$	Vertex 40: $E_1 = 0 \cdot F_{45}(x) - x = 0$ $E_4 = 1 \cdot F_{15}(x) + \frac{1}{2} \cdot (1 - F_{15}(x)) - x = 1$ $p_{14} = 0$
Vertex 41: $E_2 = 0 \cdot F_{44}(x) - x = 0$ $E_4 = 0 \cdot F_{24}(x) - x = 0$ $p_{24} = \frac{1}{2}$	Vertex 42: $E_2 = 0 \cdot F_{44}(x) - x = 0$ $E_4 = 0 \cdot F_{24}(x) - x = 0$ $p_{24} = \frac{1}{2}$
Vertex 43: $E_2 = \frac{1}{12} \cdot F_{44}(x) - x = \frac{1}{12}$ $E_4 = 0 \cdot F_{24}(x) - x = 0$ $p_{24} = 1$	Vertex 44: $E_2 = 0 \cdot F_{44}(x) - x = 0$ $E_4 = \frac{2}{3} \cdot F_{24}(x) - x = \frac{2}{3}$ $p_{24} = 0$
Vertex 45: $E_2 = \frac{2}{3} \cdot F_{44}(x) - x = \frac{2}{3}$ $E_4 = 0 \cdot F_{24}(x) - x = 0$ $p_{24} = 1$	Vertex 46: $E_2 = 0 \cdot F_{44}(x) - x = 0$ $E_4 = \frac{1}{12} \cdot F_{24}(x) - x = \frac{1}{12}$ $p_{24} = 0$
Vertex 47: $E_2 = 1 \cdot F_{44}(x) + \frac{1}{12} \cdot (1 - F_{44}(x)) - x = \frac{1}{3}$ $E_4 = \frac{2}{3} \cdot F_{24}(x) - x = 0$ $p_{24} = \frac{7}{11}$	Vertex 48: $E_2 = \frac{2}{3} \cdot F_{44}(x) - x = 0$ $E_4 = 1 \cdot F_{24}(x) + \frac{1}{12} \cdot (1 - F_{24}(x)) - x = \frac{1}{3}$ $p_{24} = \frac{4}{11}$

Table 1 : Players' expected payoffs and winning probabilities in Vertexes 37 – 48 of Figures 2 – 3.

Vertex 49: $E_1 = \frac{5}{6} \cdot F_{33}(x) + \frac{1}{24} \cdot (1 - F_{33}(x)) - x = \frac{1}{24}$ $E_3 = \frac{5}{6} \cdot F_{13}(x) + \frac{1}{24} \cdot (1 - F_{13}(x)) - x = \frac{1}{24}$ $p_{13} = \frac{1}{2}$	Vertex 50: $E_1 = \frac{2}{3} \cdot F_{33}(x) - x = \frac{7}{12}$ $E_3 = \frac{1}{12} \cdot F_{13}(x) - x = 0$ $p_{13} = \frac{15}{16}$
Vertex 51: $E_1 = \frac{1}{12} \cdot F_{33}(x) - x = 0$ $E_3 = \frac{2}{3} \cdot F_{13}(x) - x = \frac{7}{12}$ $p_{13} = \frac{1}{16}$	Vertex 52: $E_1 = 0 \cdot F_{33}(x) - x = 0$ $E_3 = 0 \cdot F_{13}(x) - x = 0$ $p_{13} = \frac{1}{2}$
Vertex 53: $E_3 = \frac{1}{24} \cdot F_{42}(x) - x = 0$ $E_4 = \frac{1}{24} \cdot F_{32}(x) - x = 0$ $p_{34} = \frac{1}{2}$	Vertex 54: $E_3 = \frac{7}{12} \cdot F_{42}(x) - x = \frac{95}{192}$ $E_4 = \frac{1}{6} \cdot F_{32}(x) + \frac{5}{64} \cdot (1 - F_{32}(x)) - x = \frac{5}{64}$ $p_{34} = \frac{207}{224}$
Vertex 55: $E_1 = \frac{5}{16} \cdot F_{21}(x) - x = \frac{1615}{5376}$ $E_2 = \frac{275}{5376} \cdot F_{11}(x) + \frac{5}{128}(1 - F_{11}(x)) - x = \frac{5}{128}$ $p_{12} = \frac{659}{672}$	

Table 1 : Players$'$ expected payoffs and winning probabilities in Vertexes 49 – 55 of Figures 2 – 3.

Vertex 1: $E_1 = 1 \cdot F_{46}(x) + \frac{1}{2} \cdot (1 - F_{46}(x)) - x = 1$ $E_4 = 0 \cdot F_{16}(x) - x = 0$ $p_{14} = 1$	Vertex 2: $E_1 = 1 \cdot F_{46}(x) + \frac{1}{2} \cdot (1 - F_{46}(x)) - x = 1$ $E_4 = 0 \cdot F_{16}(x) - x = 0$ $p_{14} = 1$
Vertex 3: $E_1 = 1 \cdot F_{46}(x) + \frac{1}{2} \cdot (1 - F_{46}(x)) - x = \frac{1}{2}$ $E_4 = \frac{1}{2} \cdot F_{16}(x) - x = 0$ $p_{14} = \frac{1}{2}$	Vertex 4: $E_1 = 1 \cdot F_{46}(x) + \frac{1}{3} \cdot (1 - F_{46}(x)) - x = \frac{2}{3}$ $E_4 = \frac{1}{3} \cdot F_{16}(x) - x = 0$ $p_{14} = \frac{3}{4}$
Vertex 5: $E_1 = \frac{1}{3} \cdot F_{46}(x) - x = \frac{1}{3}$ $E_4 = 0 \cdot F_{16}(x) - x = 0$ $p_{14} = 1$	Vertex 6: $E_1 = \frac{1}{2} \cdot F_{46}(x) - x = 0$ $E_4 = \frac{1}{2} \cdot F_{16}(x) - x = 0$ $p_{14} = \frac{1}{2}$
Vertex 7: $E_1 = 1 \cdot F_{46}(x) + \frac{1}{3} \cdot (1 - F_{46}(x)) - x = \frac{2}{3}$ $E_4 = \frac{1}{3} \cdot F_{16}(x) - x = 0$ $p_{14} = \frac{3}{4}$	Vertex 8: $E_1 = 1 \cdot F_{46}(x) + \frac{1}{2} \cdot (1 - F_{46}(x)) - x = \frac{1}{2}$ $E_4 = \frac{1}{2} \cdot F_{16}(x) - x = 0$ $p_{14} = \frac{1}{2}$
Vertex 9: $E_1 = 1 \cdot F_{46}(x) - x = 0$ $E_4 = 1 \cdot F_{16}(x) - x = 0$ $p_{14} = \frac{1}{2}$	Vertex 10: $E_1 = 1 \cdot F_{46}(x) - x = 0$ $E_4 = 1 \cdot F_{16}(x) - x = 0$ $p_{14} = \frac{1}{2}$
Vertex 11: $E_1 = \frac{1}{2} \cdot F_{46}(x) - x = 0$ $E_4 = \frac{1}{2} \cdot F_{16}(x) - x = 0$ $p_{14} = \frac{1}{2}$	Vertex 12: $E_1 = \frac{1}{2} \cdot F_{46}(x) - x = 0$ $E_4 = \frac{1}{2} \cdot F_{16}(x) - x = 0$ $p_{14} = \frac{1}{2}$

Table 2 : Players' expected payoffs and winning probabilities in Vertexes 1 – 12 of Figure 4

Vertex 13: $E_1 = \frac{1}{2} \cdot F_{46}(x) - x = 0$ $E_4 = 1 \cdot F_{16}(x) + \frac{1}{2} \cdot (1 - F_{16}(x)) - x = \frac{1}{2}$ $p_{14} = \frac{1}{2}$	Vertex 14: $E_1 = \frac{1}{3} \cdot F_{46}(x) - x = 0$ $E_4 = 1 \cdot F_{16}(x) + \frac{1}{3} \cdot (1 - F_{16}(x)) - x = \frac{2}{3}$ $p_{14} = \frac{1}{4}$
Vertex 15: $E_1 = \frac{1}{3} \cdot F_{46}(x) - x = \frac{1}{3}$ $E_4 = 0 \cdot F_{16}(x) - x = 0$ $p_{14} = 1$	Vertex 16: $E_1 = \frac{1}{2} \cdot F_{46}(x) - x = 0$ $E_4 = \frac{1}{2} \cdot F_{16}(x) - x = 0$ $p_{14} = \frac{1}{2}$
Vertex 17: $E_1 = \frac{1}{2} \cdot F_{46}(x) - x = 0$ $E_4 = \frac{1}{2} \cdot F_{16}(x) - x = 0$ $p_{14} = \frac{1}{2}$	Vertex 18: $E_1 = 0 \cdot F_{46}(x) - x = 0$ $E_4 = \frac{1}{3} \cdot F_{16}(x) - x = \frac{1}{3}$ $p_{14} = 0$
Vertex 19: $E_1 = \frac{1}{2} \cdot F_{46}(x) - x = 0$ $E_4 = \frac{1}{2} \cdot F_{16}(x) - x = 0$ $p_{14} = \frac{1}{2}$	Vertex 20: $E_1 = \frac{1}{3} \cdot F_{46}(x) - x = 0$ $E_4 = 1 \cdot F_{16}(x) + \frac{1}{3} \cdot (1 - F_{16}(x)) - x = \frac{2}{3}$ $p_{14} = \frac{1}{4}$
Vertex 21: $E_1 = \frac{1}{2} \cdot F_{46}(x) - x = 0$ $E_4 = 1 \cdot F_{16}(x) + \frac{1}{2} \cdot (1 - F_{16}(x)) - x = \frac{1}{2}$ $p_{14} = \frac{1}{2}$	Vertex 22: $E_1 = 0 \cdot F_{46}(x) - x = 0$ $E_4 = \frac{1}{3} \cdot F_{16}(x) - x = \frac{1}{3}$ $p_{14} = 0$
Vertex 23: $E_1 = 0 \cdot F_{46}(x) - x = 0$ $E_4 = 1 \cdot F_{16}(x) + \frac{1}{2} \cdot (1 - F_{16}(x)) - x = 1$ $p_{14} = 0$	Vertex 24: $E_1 = 0 \cdot F_{46}(x) - x = 0$ $E_4 = 1 \cdot F_{16}(x) + \frac{1}{2} \cdot (1 - F_{16}(x)) - x = 1$ $p_{14} = 0$

Table 2 : Players$'$ expected payoffs and winning probabilities in Vertexes 13 – 24 of Figures 4 – 5

Vertex 25: $E_2 = 0 \cdot F_{35}(x) - x = 0$ $E_3 = 0 \cdot F_{25}(x) - x = 0$ $p_{23} = \frac{1}{2}$	Vertex 26: $E_2 = 0 \cdot F_{35}(x) - x = 0$ $E_3 = \frac{1}{12} \cdot F_{25}(x) - x = \frac{1}{12}$ $p_{23} = 0$
Vertex 27: $E_2 = \frac{1}{3} \cdot F_{35}(x) - x = 0$ $E_3 = 1 \cdot F_{25}(x) + \frac{1}{3} \cdot (1 - F_{25}(x)) - x = \frac{2}{3}$ $p_{23} = \frac{1}{4}$	Vertex 28: $E_2 = 0 \cdot F_{35}(x) - x = 0$ $E_3 = 1 \cdot F_{25}(x) + \frac{1}{2} \cdot (1 - F_{25}(x)) - x = 1$ $p_{23} = 0$
Vertex 29: $E_2 = \frac{1}{12} \cdot F_{35}(x) - x = \frac{1}{12}$ $E_3 = 0 \cdot F_{25}(x) - x = 0$ $p_{23} = 1$	Vertex 30: $E_2 = 0 \cdot F_{35}(x) - x = 0$ $E_3 = 0 \cdot F_{25}(x) - x = 0$ $p_{23} = \frac{1}{2}$
Vertex 31: $E_2 = \frac{1}{2} \cdot F_{35}(x) - x = 0$ $E_3 = \frac{1}{2} \cdot F_{25}(x) - x = 0$ $p_{23} = \frac{1}{2}$	Vertex 32: $E_2 = 0 \cdot F_{35}(x) - x = 0$ $E_3 = \frac{1}{12} \cdot F_{25}(x) - x = \frac{1}{12}$ $p_{23} = 0$
Vertex 33: $E_2 = 1 \cdot F_{35}(x) + \frac{1}{3} \cdot (1 - F_{35}(x)) - x = \frac{2}{3}$ $E_3 = \frac{1}{3} \cdot F_{25}(x) - x = 0$ $p_{23} = \frac{3}{4}$	Vertex 34: $E_2 = \frac{1}{2} \cdot F_{35}(x) - x = 0$ $E_3 = \frac{1}{2} \cdot F_{25}(x) - x = 0$ $p_{23} = \frac{1}{2}$
Vertex 35: $E_2 = 1 \cdot F_{35}(x) - x = 0$ $E_3 = 1 \cdot F_{25}(x) - x = 0$ $p_{23} = \frac{1}{2}$	Vertex 36: $E_2 = \frac{1}{3} \cdot F_{35}(x) - x = 0$ $E_3 = 1 \cdot F_{25}(x) + \frac{1}{3} \cdot (1 - F_{25}(x)) - x = \frac{2}{3}$ $p_{23} = \frac{1}{4}$

Table 2 : Players′ expected payoffs and winning probabilities in Vertexes 25 – 36 of Figures 4 – 5

Vertex 37: $E_2 = 1 \cdot F_{35}(x) + \frac{1}{2} \cdot (1 - F_{35}(x)) - x = 1$ $E_3 = 0 \cdot F_{25}(x) - x = 0$ $p_{23} = 1$	Vertex 38: $E_2 = \frac{1}{12} \cdot F_{35}(x) - x = \frac{1}{12}$ $E_3 = 0 \cdot F_{25}(x) - x = 0$ $p_{23} = 1$
Vertex 39: $E_2 = 1 \cdot F_{35}(x) + \frac{1}{3} \cdot (1 - F_{35}(x)) - x = \frac{2}{3}$ $E_3 = \frac{1}{3} \cdot F_{25}(x) - x = 0$ $p_{23} = \frac{3}{4}$	Vertex 40: $E_2 = 0 \cdot F_{35}(x) - x = 0$ $E_3 = 0 \cdot F_{25}(x) - x = 0$ $p_{23} = \frac{1}{2}$
Vertex 41: $E_2 = 0 \cdot F_{44}(x) - x = 0$ $E_4 = 0 \cdot F_{24}(x) - x = 0$ $p_{24} = \frac{1}{2}$	Vertex 42: $E_2 = 0 \cdot F_{44}(x) - x = 0$ $E_4 = 0 \cdot F_{24}(x) - x = 0$ $p_{24} = \frac{1}{2}$
Vertex 43: $E_2 = \frac{1}{12} \cdot F_{44}(x) - x = \frac{1}{12}$ $E_4 = 0 \cdot F_{24}(x) - x = 0$ $p_{24} = 1$	Vertex 44: $E_2 = 0 \cdot F_{44}(x) - x = 0$ $E_4 = \frac{2}{3} \cdot F_{24}(x) - x = \frac{2}{3}$ $p_{24} = 0$
Vertex 45: $E_2 = \frac{2}{3} \cdot F_{44}(x) - x = \frac{2}{3}$ $E_4 = 0 \cdot F_{24}(x) - x = 0$ $p_{24} = 1$	Vertex 46: $E_2 = 0 \cdot F_{44}(x) - x = 0$ $E_4 = \frac{1}{12} \cdot F_{24}(x) - x = \frac{1}{12}$ $p_{24} = 0$
Vertex 47: $E_2 = 1 \cdot F_{44}(x) + \frac{1}{12} \cdot (1 - F_{44}(x)) - x = \frac{1}{3}$ $E_4 = \frac{2}{3} \cdot F_{24}(x) - x = 0$ $p_{24} = \frac{7}{11}$	Vertex 48: $E_2 = \frac{2}{3} \cdot F_{44}(x) - x = 0$ $E_4 = 1 \cdot F_{24}(x) + \frac{1}{12} \cdot (1 - F_{24}(x)) - x = \frac{1}{3}$ $p_{24} = \frac{4}{11}$

Table 2 : Players′ expected payoffs and winning probabilities in Vertexes 37 – 48 of Figures 4 – 5

Vertex 49: $E_1 = \frac{5}{6} \cdot F_{33}(x) + \frac{1}{24} \cdot (1 - F_{33}(x)) - x = \frac{1}{24}$ $E_3 = \frac{5}{6} \cdot F_{13}(x) + \frac{1}{24} \cdot (1 - F_{13}(x)) - x = \frac{1}{24}$ $p_{13} = \frac{1}{2}$	Vertex 50: $E_1 = \frac{2}{3} \cdot F_{33}(x) - x = \frac{7}{12}$ $E_3 = \frac{1}{12} \cdot F_{13}(x) - x = 0$ $p_{13} = \frac{15}{16}$
Vertex 51: $E_1 = \frac{1}{12} \cdot F_{33}(x) - x = 0$ $E_3 = \frac{2}{3} \cdot F_{13}(x) - x = \frac{7}{12}$ $p_{13} = \frac{1}{16}$	Vertex 52: $E_1 = 0 \cdot F_{33}(x) - x = 0$ $E_3 = 0 \cdot F_{13}(x) - x = 0$ $p_{13} = \frac{1}{2}$
Vertex 53: $E_3 = \frac{1}{24} \cdot F_{42}(x) - x = 0$ $E_4 = \frac{1}{24} \cdot F_{32}(x) - x = 0$ $p_{34} = \frac{1}{2}$	Vertex 54: $E_3 = \frac{7}{12} \cdot F_{42}(x) - x = \frac{95}{192}$ $E_4 = \frac{1}{6} \cdot F_{32}(x) + \frac{5}{64} \cdot (1 - F_{32}(x)) - x = \frac{5}{64}$ $p_{34} = \frac{207}{224}$
Vertex 55: $E_1 = \frac{5}{16} \cdot F_{21}(x) - x = \frac{1615}{5376}$ $E_2 = \frac{275}{5376} \cdot F_{11}(x) + \frac{5}{128}(1 - F_{11}(x)) - x = \frac{5}{128}$ $p_{12} = \frac{659}{672}$	

Table 2 : Players′ expected payoffs and winning probabilities in Vertexes 49 – 55 of Figures 4 – 5