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Round-Robin Tournaments with a Dominant Player

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Abstract

We analyze the subgame perfect equilibrium of the round-robin tournament with one strong (dominant) and two weak players, and compare between this tournament and the one-stage contest with respect to the players' expected payoffs, expected total effort and their probabilities to win. We find that if the designer's goal is to maximize the dominant player's probability to win then he should use the round-robin tournament given that the weak players are matched in the second stage. If, however, the allocation of players is randomly determined then the contest designer might prefer the one-stage contest. Last, if the contest designer's goal is to maximize the players' expected total effort, then if the asymmetry between the players is relatively low he should use the one-stage contest, but, if the asymmetry is relatively high the round-robin tournament should be used.

JEL Classifications: D44, O31

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1 Introduction

Tournaments are prevalent in many areas of life including labor markets (Lazear and Rosen 1981, Prendergast 1999), political races (Klumpp and Polborn 2006), R&D (Harris and Vickers 1985, Zizzo 2002, Breitmoser et al. 2010), rent seeking (Tullock 1980, Gradstein and Konrad 1999), sports (Rosen 1986, Szymanski 2003, Harbaugh and Klumpp 2005), etc. In this paper we focus on the round-robin tournaments in which every player or team competes against all the others, and in every stage a player plays a pair-wise match against a different opponent. Such particular tournaments commonly take place in professional football and basketball leagues but sometimes can be seen in other domains. To illustrate, in the 2015 elections for Israel's Knesset a representative of each party was invited for a TV debate which was organized as a round-robin tournament where in each stage the parties' representatives were divided into different pairs, with each pair confronting each other for several minutes.

In the literature on contests, the most common goal, especially in sport contests, is to maximize the players' total effort (Szymanski 2003). However, the contest designer may also want to increase the competitive balance by decreasing the differences among the players' probabilities of winning, or, alternatively, he may want to affect the identity of the winner by determining the players' probabilities of winning (Groh et al. 2012). This can be done by choosing the type of tournament. We address these issues by comparing between the round-robin tournament and the standard one-stage contest in which all the players compete against each other only once in one grand contest. The comparison is done with respect to the players' expected payoffs, their probabilities of winning, and their expected total effort. It is important to note that when the players are asymmetric the results of the round-robin tournament depend on the allocations of players in the different stages of the tournament. Thus, since the number of different allocations grows exponentially in the number of players, we focus on the simple case of three players where one is dominant, i.e., one player has a higher value of winning than the other (weaker) players.[1] Over the years this format of round-robin tournaments with three players has been used in many Olympic Games tournaments such as wrestling, badminton, women's soccer, etc. It was also used in several geographical zones of the FIFA World Cup qualifications. In our round-robin tournament there are three possible allocations of the players,

[1] We believe that some of our main results hold for round-robin tournaments with any number of players.

all of which are considered in this paper. For both types of contests, each match is modeled as an all-pay contest.[2] In the all-pay contest (auction) the contestant with the highest effort (output) wins the contest, but all the contestants bear the cost of their effort.[3]

We find that, independent of the allocation of players, the expected payoffs of the weak players in the round-robin tournament is higher than or equal to their expected payoffs in the one-stage contest. In contrast, depending on the players' allocation in the round-robin tournament, the expected payoff of the dominant player in the one-stage contest can be either higher or lower than in the round-robin tournament. Furthermore, if the dominant player is allocated in the first and the last stages of the round-robin tournament his expected payoff in the round-robin tournament is higher than in the one-stage contest. The intuitive explanation for this is that if the dominant player wins in the first stage his expected value of winning increases and his opponents' expected values of winning decreases in the following stages. In addition, by playing in the last stage the dominant player might compete against an opponent that lost in the previous stages and therefore has a low chance to win the tournament. This situation enables the dominant player to win in the last stage and particularly to win the entire tournament without exerting much effort. Therefore, allocation in the first and last stages is favorable for the dominant player and in that case he prefers the round-robin tournament over the one-stage contest.

However, usually, a player cannot choose the stages in which he is allocated and actually the allocation of players in the round-robin tournament is randomly determined, namely, each possible allocation of players is chosen by the same probability. In that case, the dominant player's expected payoff in the round-robin tournament will be higher than in the one-stage contest given that the asymmetry between the players is relatively low and vice versa if the asymmetry is high. Thus, while the weak players prefer the round-robin tournament, the dominant player does not necessarily prefer either of the contests.

Using this analysis of the subgame perfect equilibrium, we then calculate the dominant player's probability

[2] Numerous applications of the all-pay contest have been made to rent-seeking and lobbying in organizations, R&D races, political contests, promotions in labor markets, trade wars, military and biological wars of attrition (see, for example, Che and Gale (1998) and Moldovanu and Sela (2001)).

[3] The all-pay contest is the limit point of the popular Tullock contest with the success function $p_i(x_1, x_2) = \frac{(x_i)^r}{(x_1)^y + (x_2)^r}, i = 1, 2$ when r converges to infinity. Thus, we can conjecture that our results hold for the Tullock success function when r is sufficiently large.

to win in the round-robin tournament and then show by numerical analysis that, independent of the level of asymmetry, the dominant player's probability to win is highest when the weak players are matched in the second stage. The reason again is, if the dominant player wins in the first stage he dramatically increases the difference in his expected value of winning and his opponents' values of winning in the next stages and accordingly he increases his probability to win his next game. On the other hand, if the dominant player does not play in the first stage he might play against a weak player who already won in the previous stage and therefore will have an expected value that is higher than his own, i.e., the dominant player will no longer be dominant. Thus, if a contest designer wishes to maximize the dominant player's probability to win, he should organize a round-robin tournament and allocate the dominant player in the first and last stages. However, if the players are randomly allocated in the round-robin tournament, independent of the asymmetry of the players, the dominant player's probability to win in the one-stage contest is higher than in the round-robin tournament when the players are randomly allocated. These findings indicate that the common intuition according to which the dominant player's probability to win is always higher in the round-robin tournament than in the one-stage contest is not correct.

We also calculate the players' expected total effort and compare it between the round-robin tournament and the one-stage contest. Our results show that for every allocation of players in the round-robin tournament, in particular when the allocation is randomly determined, if the asymmetry is relatively low, the expected total effort is higher in the one-stage contest, while if the asymmetry is relatively high, then the expected total effort is higher in the round-robin tournament. The reason for these results is that when the asymmetry between the players is relatively high by allocating the weak players in the first stage, the designer gives them advantage as we explained above, the tournament becomes more balanced and then the players exert more effort than in the one-stage contest. Hence, if the contest designer wishes to maximize the players' expected total effort whether he chooses the round-robin tournament or the one-stage contest will depend on the asymmetry between the players.

It should be mentioned that, independent of the level of asymmetry, the expected total effort is minimized when the two weak players are matched in the second stage of the round-robin tournament while the dominant player's probability to win is maximized when the two weak players are matched in the second stage of the round-robin tournament. Thus, a contest designer will not be able to maximize the expected total effort together with the dominant player's probability of winning the tournament. The reason for these findings is very simple since, again, the more balanced the contest is the higher will be the players' expected total effort.

We focus on the comparison between the round-robin tournament and the one-stage contest, but it is important to emphasize that the analysis of the one-stage contest with two weak players and one dominant player is strategically equivalent to an elimination tournament in which the two weak players usually compete in the semifinal and the winner competes against the dominant player in the final. In the elimination tournament the expected total effort in the semifinal approaches zero and each of the weak players plays in the final with the same probability. Thus, our comparison between the one-stage contest and the round-robin tournament is equivalent to the comparison between the elimination tournament and the round-robin tournament. Similarly to Groh et al. (2012) who studied optimal seeding in elimination tournaments, in our round-robin model we assume that the winning probabilities in each match are endogenous in that they result from mixed equilibrium strategies and are positively correlated to winning valuations. Furthermore, like Groh et al. (2012) the winning probabilities depend on the stage of the tournament where the match takes place as well as on the identity of the future expected opponents.[4]

Our paper is related to several other works that focus on the importance of the first and the last stages in multi-stage contests. For example, Klumpp and Polborn (2006) showed that in sequential elections between two candidates, the loser of the first district will have a lower incentive to exert a costly effort in the second district than the winner of the first district. This yields an increased probability of the winner of the first district to win again in the second district. Deck and Sheremeta (2012) showed that if a defender wins early battles in the game of siege, the attacker becomes discouraged and as a result the probability for him to win any future battles decreases. Page and Page (2007) found empirically that in the best-of-two European soccer cup competitions, the second leg home team has more than a 50% probability to win. Krumer (2013)

[4] The statistical literature on the design of various forms of tournaments (see, David (1959), Glenn (1960) and Searles (1963)) assumes that for each match among players i and j there is a fixed, exogenously given probability that i beats j. Thus, in contrast to Groh et al. (2012), this probability does not depend on the stage of the tournament where the particular match takes place nor on the identity of the expected opponent at the next stage.

explained this finding by revealing a possible psychological advantage to the winner of the first stage.

The rest of the paper is organized as follows: Section 2 presents the equilibrium analysis of the one-stage contest. Section 3 presents the subgame perfect equilibrium in the three possible allocations of players in the round-robin tournament with a dominant player and two weak players. In Section 4 we analyze the players' expected payoffs in the round-robin tournament and compare them to those in the one-stage contest. In Sections 5 and 6 we compare the dominant player's probability of winning and the players' expected total effort between the one-stage contest and the round-robin tournament. Appendixes A, B and C include a complete analysis of the subgame perfect equilibrium for the different allocations of players in the round-robin tournament. Appendix D (an online appendix) calculates the dominant player's probability of winning and the players' expected total effort in the round-robin tournament.

2 The one-stage (all-pay) contest

We begin with the analysis of the standard one-stage (all-pay) contest which will serve as a benchmark for the round-robin (all-pay) tournament. In the one-stage contest with a single prize the player with the highest effort (output) wins the entire prize, but all the players bear the cost of their effort. Consider a one-stage contest with three players $1, 2, 3$ where the players' values of winning are $v = v_3 \geq v_1 = v_2 = 1$. According to Hillman and Riley (1989) and Baye, Kovenock and de Vries (1996), there is always a mixed-strategy equilibrium in which players 3 and 2 (or alternatively, players 3 and 1) randomize on the interval $[0, 1]$ according to their effort cumulative distribution functions which are given by

$$\begin{aligned} v \cdot F_2(x) - x &= v - 1 \\ 1 \cdot F_3(x) - x &= 0 \end{aligned} \tag{1}$$

We can see that the expected payoff of player 2 (and player 1) is zero, while player 3's is $v - 1$. Player 3's effort is distributed according to the cumulative distribution function

$$F_3(x) = x$$

while player 2's (or 1's) effort is distributed according to the cumulative distribution function

$$F_2(x) = \frac{v-1+x}{v}$$

Given these mixed strategies, player 3's winning probability against player 2 (or player 1) is

$$p_3 = 1 - \frac{1}{2v} \tag{2}$$

If we assume that players 1 and 2 have the same probability to be the opponent who plays against player 3, then each of them has the following probability of winning the contest

$$p_1 = p_2 = \frac{1}{2} \cdot \frac{1}{2v} = \frac{1}{4v} \tag{3}$$

The players' expected total effort is given by

$$TE = \frac{1}{2} \cdot (1 + \frac{1}{v}) = \frac{v+1}{2v} \tag{4}$$

Alternatively, consider the situation in which these three players compete in an elimination tournament where the two weak players (1 and 2) simultaneously compete in the semifinal (first stage) and the winner competes in the final (second stage) against the dominant player (player 3).[5] Then the winner of the final wins the tournament and obtains the prize. In the subgame perfect equilibrium of this elimination tournament, players 1 and 2 do not exert any effort in the first stage, and each of them plays against player 3 in the second stage with the same probability. Hence, we can see that the elimination tournament with two weak players and one dominant player is exactly equivalent to the one-stage contest. Using the above analysis of the one-stage (all-pay) contest we can now turn to analyze the players' equilibrium strategies in the round-robin tournament and, in particular, to compare the players' performances in both contest types.

3 The round-robin (all-pay) tournament

Consider three players (or teams) $i = 1, 2, 3$ competing for a single prize. We assume that there are two weak players (players 1 and 2) who have the same value of winning $v_1 = v_2 = 1$ and a dominant player

[5] This natural allocation of players in the elimination tournament with three players increases the dominant player's probability to win.

(player 3) who has a higher value of winning $v_3 = v > 1$. The players' valuations are common knowledge and a player's cost is $c(x_i) = x_i$ where x_i is his effort. The players play pair-wise matches and we model each game between two players as an all-pay auction where both players simultaneously exert efforts, and the player with the higher effort wins the game. In the round-robin tournament the players compete one time against each other in sequential games, such that every player plays two games. A player who wins two games wins the tournament. In the case that each player wins only once, there will be a draw to determine the winner. If one of the players wins in the first two stages, the winner of the tournament is decided and the game in the third stage is not played. In our round robin tournament with two weak players and one dominant player there are three possible allocations of players as follows:

1) **Case A:** The weak players are matched in the first stage. Then, the order of the games is

Stage 1:	Game 1: player 1 - player 2
Stage 2:	Game 2: player 1 - player 3
Stage 3:	Game 3: player 2 - player 3

2) **Case B:** The weak players are matched in the second stage. Then, the order of the games is

Stage 1:	Game 1: player 1 - player 3
Stage 2:	Game 2: player 1 - player 2
Stage 3:	Game 3: player 2 - player 3

3) **Case C:** The weak players are matched in the third stage. Then, the order of the games is

Stage 1:	Game 1: player 1 - player 3
Stage 2:	Game 2: player 2 - player 3
Stage 3:	Game 3: player 1 - player 2

Figures 1, 2 and 3 present all the possible paths of this tournament for cases A, B and C, respectively [Figures 1, 2 and 3 about here]. In the decision node F in each of these figures, two players compete in the first stage. In the decision nodes E and D two players compete in the second stage, and in the decisions nodes A, B and C, two players compete against each other in the third stage. For each decision node (A-E) there is a different path from the initial node F, namely, a different history of the games in the previous stages.

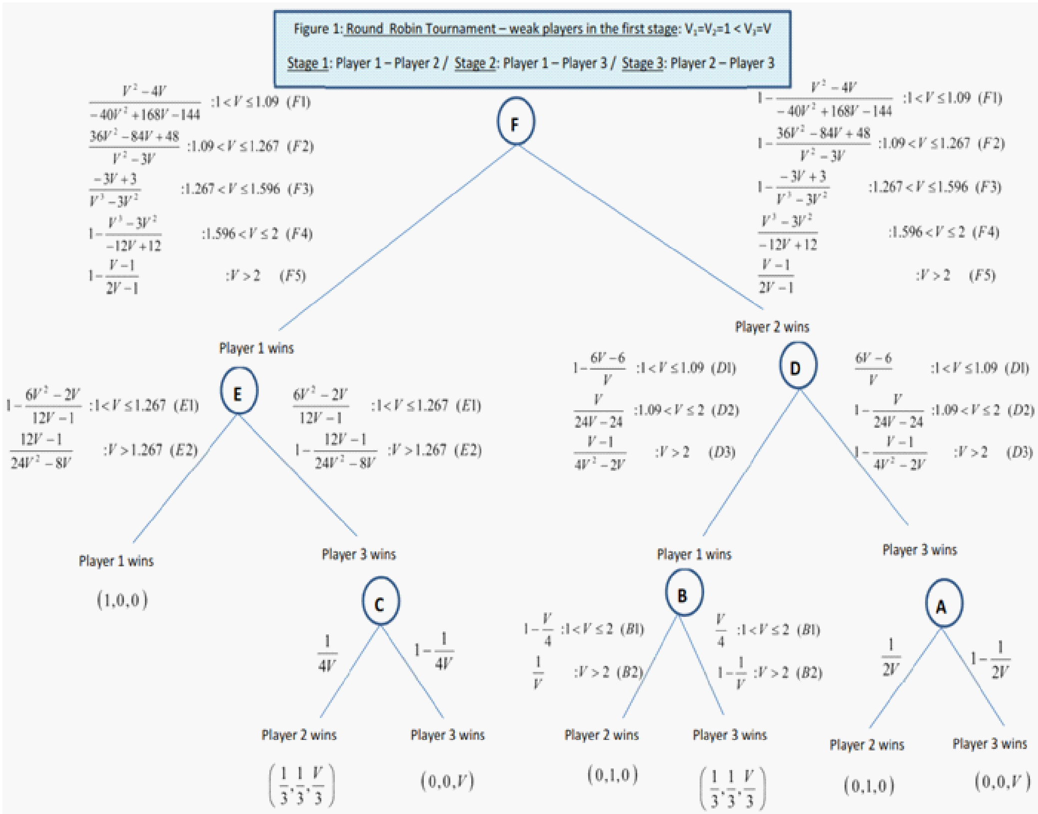

Figure 1: The possible paths of the round-robin tournament when the two weak players are matched in the first stage.

The players' payoffs are indicated in the terminal nodes. The formulas on the sides of the branches denote the winning probabilities of the players who compete in the appropriate decision nodes.

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4 The players' expected payoffs

In this section we first investigate the effect of the allocation of players in the round-robin tournament on the players' expected payoffs. Then we compare the findings to that of the one-stage contest.

Proposition 1 *In the round-robin tournament with one dominant player and two weak players, if the weak players are matched in the first or the third stage and the asymmetry between the players is relatively weak, the expected payoff of the dominant player is lower than the expected payoffs of the weak players. However,*

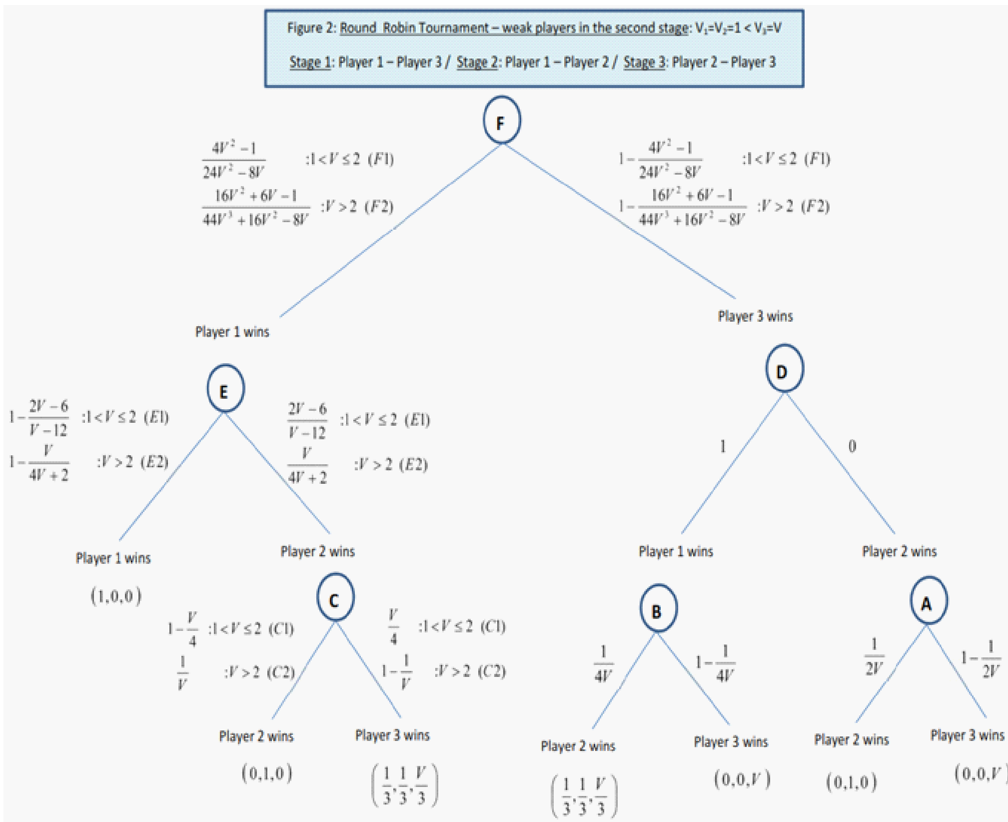

Figure 2: The possible paths of the round-robin tournament when the two weak players are matched in the second stage.

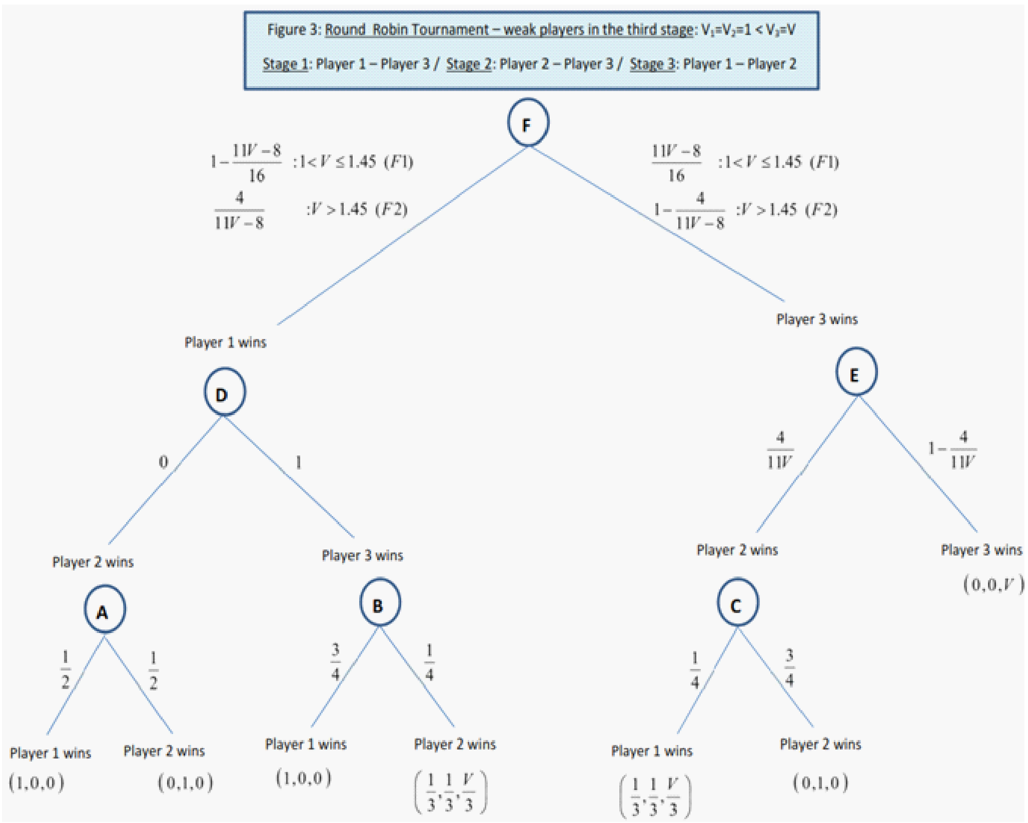

Figure 3: The possible paths of the round-robin tournament when the two weak players are matched in the third stage.

if the weak players are matched in the second stage, then independent of whether the asymmetry is weak or strong, the expected payoff of the dominant player is always higher than the expected payoffs of the weak players.

Proof. The proof is given in Appendix D (online appendix). ■

According to Proposition 1, if the dominant player can choose the players' allocation in the round-robin tournament he would allocate his two weak opponents in the second stage such that he will compete against one of them in the first stage and against the other in the last stage. The intuition behind this result is as follows: If the weak players are allocated in the first stage then the dominant player competes against the winner in the first stage with a relatively high probability. In that case, he might have a lower chance to win against his opponent in the second stage since he might have a lower expected value of winning than his opponent who needs only to win one more time in order to win the entire tournament. Thus, the dominant player might no longer be dominant and therefore such an allocation is not profitable for him. If, on the other hand, the weak players are matched in the third stage, the dominant player competes against each of his opponents in the first two stages in which each of them still has a real chance to win the tournament. Then, the weak players exert relatively high efforts in the first two stages and the dominant player will exert an even higher effort if he wants to win. Consequently, the dominant player's expected payoff is relatively low when the weak players are matched in the third stage. However, if the weak players are matched in the second stage, i.e., the dominant player plays in the first and the third stages, the dominant player has a relatively high chance to win in the first stage. Moreover, the dominant player also has a chance to compete against his opponent in the last stage when this opponent lost in his previous game. This situation enables the dominant player to win in the last stage and even win the entire tournament without exerting much effort. By comparing each player's expected payoff in the one-stage contest given by (1) with that of the round-robin tournament given in Appendix D, we obtain that

Proposition 2 *In a competition between one dominant and two weak players, the expected payoff of the weak players in the round-robin tournament is higher than or equal to their expected payoffs in the one-stage contest. On the other hand, depending on the players' allocation, the expected payoff of the dominant player could be either higher or lower than in the one-stage contest. If the allocation of the players in the round-*

robin tournament is random, namely, each allocation of players is chosen with the same probability, then the dominant player's expected payoff in the round-robin tournament is higher than in the one-stage contest when the asymmetry between the players is relatively low and vice versa if the asymmetry is high.

Intuitively, multi-stage contests like the round-robin tournament yield a higher expected payoff for the dominant player than a one-stage contest since in the round-robin tournament the dominant player can make up for an unexpected loss in one of the stages and still win the entire tournament. However, according to Proposition 2, the round-robin tournament does not necessarily yield a higher expected payoff for the dominant player than the one-stage contest, but rather is more profitable for the dominant player only if the weak players are matched in the second stage. If, on the other hand, the allocation of players is randomly determined, then, depending on whether the asymmetry is low or high, the dominant player's expected payoff might be either higher or lower than in the one-stage contest. The reason is that when the asymmetry among the players is high, by choosing an allocation of players in which the dominant player is not allocated in the first and third stages, the designer can reduce the advantage of the dominant player while in the one-stage contest this advantage remains the same.

5 The dominant player's winning probability

In this section we examine how the contest designer can maximize the dominant player's probability of winning by choosing an appropriate allocation of players. In the one-stage contest, the dominant player's probability to win is given by (2). Using the equilibrium analysis in Appendices A, B and C, we explicitly calculate in Appendix D the dominant player's probability of winning in the round-robin tournament for all three possible allocations of players. Then by numerical analysis we show in Figure 4 the dominant player's probability to win as a function of the level of asymmetry in both contest forms [Figure 4 about here].

We can see that, independent of the level of asymmetry, the dominant player's probability to win is highest in the round-robin tournament when the weak players are matched in the second stage. The intuition, as we already explained, is very simple. When the dominant player competes in the last stage he always has a chance to win especially if he competes against a weak player who lost in the previous stage. Furthermore, if the dominant player does not play in the first stage he might compete against a weak player who actually has a higher expected value of winning since he needs only one more win to be the winner of the tournament while the dominant player needs to win in the following two stages. Thus, if a contest designer wishes to maximize the dominant player's probability to win, he should organize a round-robin tournament and allocate the weak players in the second stage. However, the dominant player's probability to win is always higher in the one-stage contest than in the round-robin tournament when the weak players are not matched in the second stage. In that case, if the asymmetry is relatively low, i.e., $v < 1.03$, the dominant player's probability to win is lowest in the round-robin tournament when the weak players are matched in the first stage, but if the asymmetry is relatively high, i.e., $v > 1.03$, the dominant player's probability to win is lowest in the round-robin tournament when the weak players are matched in the third stage. As we claimed previously, these findings indicate that the standard intuition according to which the dominant player's probability to win is higher in the round-robin tournament than in the one-stage contest is not completely correct.

Figure 4 also indicates that, independent of the asymmetry of the players, the dominant player's probability to win is higher in the one-stage contest than in the round-robin tournament when the players are randomly allocated, i.e., each allocation of players is chosen with the same probability of 1/3. Hence, in such a case, if the goal is to maximize the dominant player's probability to win the tournament, the contest designer should choose a one-stage contest instead of the round-robin tournament.

6 The players' total effort

One of the possible goals of a contest designer is to maximize the expected total effort. In the round-robin tournament the designer can affect the players' expected effort by choosing the allocation of players. Using the equilibrium analysis in Appendices A, B and C, we explicitly calculate in Appendix D the expected total effort in the round-robin tournament and compare it to the expected total effort in the one-stage contest (given by (4)). Then, we show by numerical analysis in Figure 5 the expected total effort as a function of the level of asymmetry in the round-robin tournament as well as in the one-stage contest [Figure 5 about here]. We can see that in the round-robin tournament, if the asymmetry is relatively low, i.e., $v < 1.38$ the expected total effort is maximized when the two weak players (players 1 and 2) are matched in the first

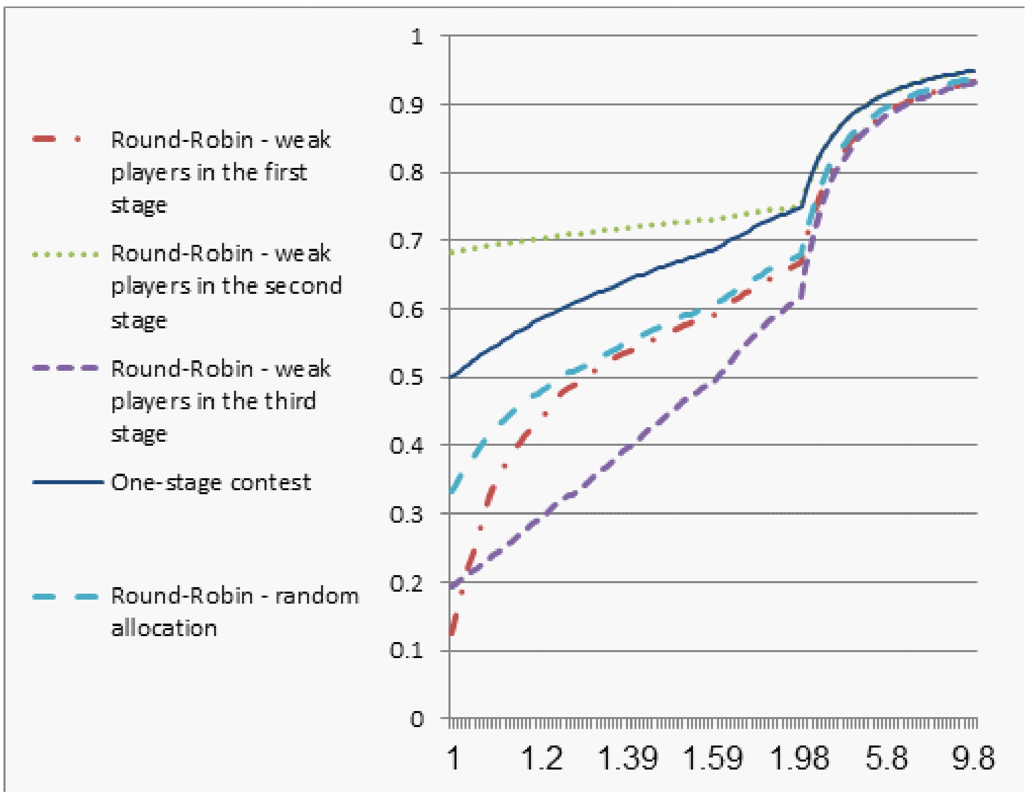

Figure 4: The dominant player's probability to win as a function of the level of asymmetry (v) for the one-stage contest and the three round-robin tournaments (the weak players are matched either in the first, second or third stage).

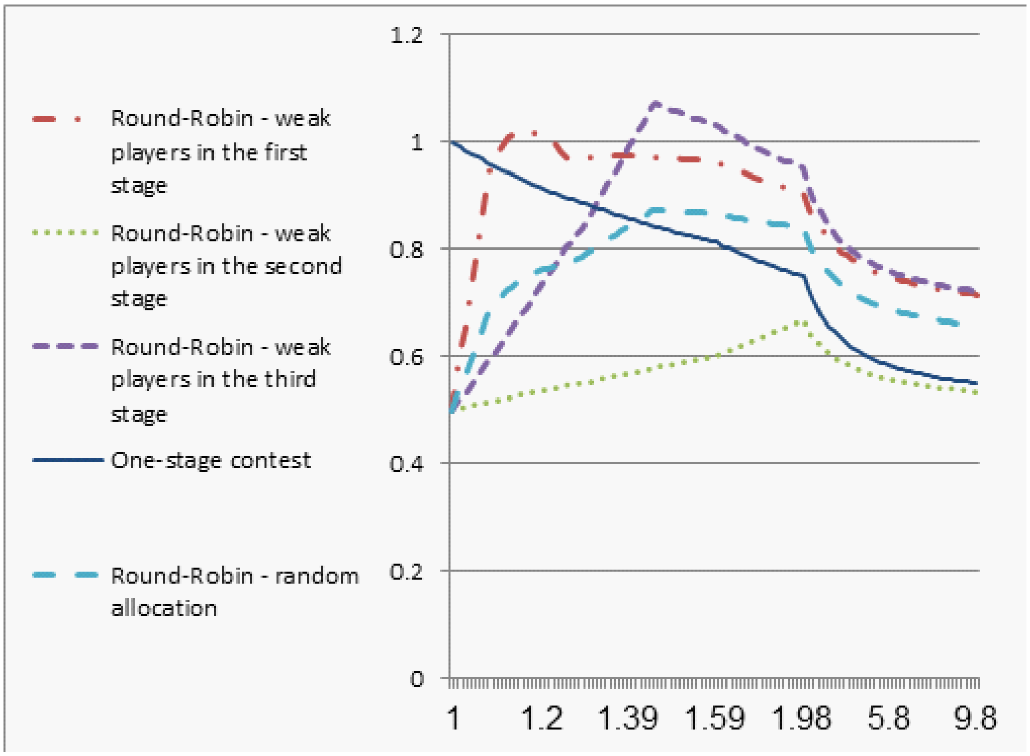

Figure 5: The expected total effort as a function of the level of asymmetry (v) for the one-stage contest and the three round-robin tournaments (the weak players are matched either in the first, second or third stage).

stage, but if the asymmetry is relatively high, i.e., $v > 1.38$ the expected total effort is maximized when the two weak players are matched in the third stage. In addition, if the asymmetry is relatively low, i.e., $v < 1.1$ the expected total effort is higher in the one-stage contest while if the asymmetry is relatively high, i.e., $v > 1.1$ the expected total effort is higher in the round-robin tournament. Moreover, independent of the level of asymmetry, the expected total effort for both contest forms is minimized when the two weak players are matched in the second stage of the round-robin tournament. The intuition for this last result is very simple since, as we showed in the previous section, when the two weak players are matched in the second stage the dominant player's probability to win is maximized. Then the competition between the dominant player and the weak players is not balanced and it is well known that when a contest is less balanced the expected effort of the players is lower (see, for example, Baye, Kovenock and de Vries (1996) and Konrad (2008)). Thus, for the round-robin tournament, if a contest designer wishes to maximize the expected total effort he should not allocate the weak players in the second stage.

It is important to emphasize that for every allocation of the players in the round-robin tournament, if the asymmetry is relatively low, the expected total effort is higher in the one-stage contest, while if the asymmetry is relatively high, the expected total effort is higher in the round-robin tournament. Thus, this relation between the one-stage contest and the round-robin tournament regarding the players' expected total effort holds even when the allocation of players in the round-robin tournament is randomly determined.

7 Concluding remarks

We studied round-robin tournaments with three players where one player is dominant, i.e., he has a higher value of winning than his weaker opponents. We demonstrated that the expected payoff of the weak players in the round-robin tournament is higher than or equal to their expected payoffs in the one-stage contest, but the expected payoff of the dominant player in the one-stage contest could be either higher or lower than in the round-robin tournament. We also showed that if a contest designer wishes to maximize the dominant player's probability to win and he can determine the allocation of players, then he should organize a round-robin tournament. But, if the allocation of players in the round-robin tournament is random, he should organize a one-stage contest. In addition, if the contest designer wishes to maximize the players' expected total effort, then if the asymmetry between the players is relatively low, the one-stage contest is preferred, while if the asymmetry is relatively high, the round-robin tournament should be chosen. It would be interesting to extend the analysis to more than three asymmetric players. However, such an extension is not simple due to the complex combinatorial structure of the round-robin tournament.

8 Appendix A: The weak players are matched in the first stage of the round-robin tournament

In order to analyze the subgame perfect equilibrium of the round robin tournament we begin with the third stage of the contest and go backwards to the first stage. First we analyze the case where the weak players (players 1 and 2) are matched in the first stage. The possible paths of this tournament are described by Figure 1. We denote by $p_{i,j}^{*}$ the probability that player i wins against player j in node * of the tree-game.

8.1 Stage 3 (player 2 vs. player 3)

We have the following three scenarios:

1) Assume first that player 2 won the game in the first stage and player 3 won in the second stage (node A in Figure 1). Since each of these players (2 and 3) won once before the last stage, the winner of the game between them in the third stage wins the entire tournament. In that case player 2's payoff would be 1, and player 3's payoff would be v. The loser's payoff in that stage will be zero. Thus, in the unique mixed strategy equilibrium the players randomize on the interval $[0, 1]$ according to their effort cumulative distribution functions $F_i^3, i = 2, 3$ which are given by

$$\begin{aligned} 1 \cdot F_3^3(x) - x &= 0 \\ v \cdot F_2^3(x) - x &= v - 1 \end{aligned} \tag{5}$$

Then, player 2's winning probability against player 3 in the third stage (node A) is given by

$$p_{23}^A = \frac{1}{2v} \tag{6}$$

and the expected total effort in the third stage (node A) is given by

$$TE^A = \frac{1}{2} \cdot (1 + \frac{1}{v}) = \frac{v+1}{2v} \tag{7}$$

2) Assume now that player 2 won the game in the first stage and player 3 lost the game in the second stage (node B in Figure 1). Thus, if player 2 also wins in the third stage he wins the tournament. In that case, his payoff is 1, whereas player 3's payoff is zero. But, if player 3 wins in this stage, then every player wins once and a draw will determine the winner of the entire tournament. Then, players 1 and 2's payoff is equal to $\frac{1}{3}$ and player 3's payoff is equal to $\frac{v}{3}$.

In this case we have to consider two different subcases of asymmetry, i.e. $1 < v \leq 2$ and $v > 2$. If $1 < v \leq 2$ (node B1 in Figure 1), there is a unique mixed strategy equilibrium in which players 2 and 3 randomize on the interval $[0, \frac{v}{3}]$ according to their effort cumulative distribution functions $F_i^3, i = 2, 3$ which are given by

$$
\begin{aligned}
1 \cdot F_3^3(x) + \frac{1}{3} \cdot (1 - F_3^3(x)) - x &= 1 - \frac{v}{3} \\
\frac{v}{3} \cdot F_2^3(x) - x &= 0
\end{aligned}
\tag{8}
$$

Then, player 2's winning probability against player 3 in the third stage (node B1 in Figure 1) is given by

$$p_{23}^{B1} = 1 - \frac{v}{4} \tag{9}$$

and the expected total effort in the third stage (node B1 in Figure 1) is given by

$$TE^{B1} = \frac{v}{6} \cdot (1 + \frac{v}{2}) = \frac{v^2 + 2v}{12} \tag{10}$$

If, however, $v > 2$ (node B2 in Figure 1), there is a unique mixed strategy equilibrium in which players 2 and 3 randomize on the interval $[0, \frac{2}{3}]$ according to their effort cumulative distribution functions $F_i^3, i = 2, 3$ which are given by

$$
\begin{aligned}
1 \cdot F_3^3(x) + \frac{1}{3} \cdot (1 - F_3^3(x)) - x &= \frac{1}{3} \\
\frac{v}{3} \cdot F_2^3(x) - x &= \frac{v - 2}{3}
\end{aligned}
\tag{11}
$$

Then, player 2's winning probability against player 3 in the third stage (node B2 in Figure 1) is given by

$$p_{23}^{B2} = \frac{1}{v} \tag{12}$$

and the expected total effort in the third stage (node B2 in Figure 1) is given by

$$TE^{B2} = \frac{1}{3} \cdot (1 + \frac{2}{v}) = \frac{v + 2}{3v} \tag{13}$$

3) We now assume that player 2 lost in the first stage and player 3 won in the second stage (node C in Figure 1). Thus, if player 3 also wins in the third stage he wins the entire tournament. Then his payoff is v, whereas player 2's payoff is zero. But if player 2 wins in this stage, then every player wins only once and a draw will determine the winner of the entire tournament. In that case, players 1 and 2's payoffs are equal to $\frac{1}{3}$ and player 3's payoff is equal to $\frac{v}{3}$. Thus, in the unique mixed strategy equilibrium players 2 and 3

randomize on the interval $[0, \frac{1}{3}]$ according to their effort cumulative distribution functions $F_i^3, i = 2, 3$ which are given by

$$\begin{aligned} \frac{1}{3} \cdot F_3^3(x) - x &= 0 \\ v \cdot F_2^3(x) + \frac{v}{3} \cdot (1 - F_2^3(x)) - x &= v - \frac{1}{3} \end{aligned} \tag{14}$$

Then, player 2's winning probability against player 3 in the third stage (node C in Figure 1) is given by

$$p_{23}^C = \frac{1}{4v} \tag{15}$$

and the expected total effort in the third stage (node C in Figure 1) is given by

$$TE^C = \frac{1}{6} \cdot (1 + \frac{1}{2v}) = \frac{2v + 1}{12v} \tag{16}$$

8.2 Stage 2 (player 1 vs. player 3)

We have two possible scenarios:

1) Assume that player 1 lost in the first stage (node D in Figure 1). In that case, if player 1 wins in the second stage and player 3 wins in the third stage, then player 1's expected payoff is $\frac{1}{3}$; otherwise, player 1's expected payoff is zero. However, if player 3 wins in the second stage, by (5), his expected payoff in the next stage is $v - 1$. But if he loses, three different subcases of asymmetry arise, i.e., $1 < v \leq 1.09$, $1.09 < v \leq 2$ and $v > 2$. First, assume that $1 < v \leq 1.09$ (node D1 in Figure 1). If player 3 loses in the second stage, by (8), his expected payoff in the next stage is zero. Then, by (5), (8) and (9), there is a unique mixed strategy equilibrium in which players 1 and 3 randomize on the interval $[0, v - 1]$ according to their effort cumulative distribution functions $F_i^2, i = 1, 3$ which are given by

$$\begin{aligned} (\frac{v}{4} \cdot \frac{1}{3}) \cdot F_3^2(x) - x &= 1 - \frac{11v}{12} \\ (v - 1) \cdot F_1^2(x) - x &= 0 \end{aligned} \tag{17}$$

Then, player 1's winning probability against player 3 in the second stage (node D1 in Figure 1) is given by

$$p_{13}^{D1} = 1 - \frac{6v - 6}{v} \tag{18}$$

and the expected total effort in the second stage (node D1 in Figure 1) is given by

$$TE^{D1} = (\frac{v - 1}{2}) \cdot (1 + \frac{12v - 12}{v}) = \frac{13v^2 - 25v + 12}{2v} \tag{19}$$

Now, assume that $1.09 < v \leq 2$ (node D2 in Figure 1). Then, as previously, by (5), (8) and (9), there is a unique mixed strategy equilibrium in which players 1 and 3 randomize on the interval $[0, \frac{v}{12}]$ according to their effort cumulative distribution functions $F_i^2, i = 1, 3$ which are given by

$$\begin{aligned} (\frac{v}{4} \cdot \frac{1}{3}) \cdot F_3^2(x) - x &= 0 \\ (v-1) \cdot F_1^2(x) - x &= \frac{11v}{12} - 1 \end{aligned} \tag{20}$$

Then, player 1's winning probability against player 3 in the second stage (node D2 in Figure 1) is given by

$$p_{13}^{D2} = \frac{v}{24v - 24} \tag{21}$$

and the expected total effort in the second stage (node D2 in Figure 1) is given by

$$TE^{D2} = \frac{v}{24} \cdot (1 + \frac{v}{12v - 12}) = \frac{13v^2 - 12v}{288v - 288} \tag{22}$$

Finally, assume that $v > 2$ (node D3 in Figure 1). In that case, as previously, if player 3 wins, by (5), his expected payoff in the next stage is $v - 1$. But if he loses, by (11), his expected payoff in the next stage is $\frac{v-2}{3}$. If player 1 wins in that stage, and player 3 wins in the third stage, player 1's expected payoff is $\frac{1}{3}$; otherwise, player 1's expected payoff is zero. Then, by (5), (11) and (12) there is a unique mixed strategy equilibrium in which players 1 and 3 randomize on the interval $[0, \frac{v-1}{3v}]$ according to their effort cumulative distribution functions $F_i^2, i = 1, 3$ which are given by

$$\begin{aligned} (1 - \frac{1}{v}) \cdot \frac{1}{3} \cdot F_3^2(x) - x &= 0 \\ (v-1) \cdot F_1^2(x) + (\frac{v-2}{3}) \cdot (1 - F_1^2(x)) - x &= \frac{3v^2 - 4v + 1}{3v} \end{aligned} \tag{23}$$

Then, player 1's winning probability against player 3 in the second stage (node D3 in Figure 1) is given by

$$p_{13}^{D3} = \frac{v - 1}{4v^2 - 2v} \tag{24}$$

and the expected total effort in the second stage (node D3 in Figure 1) is given by

$$TE^{D3} = (\frac{v-1}{6v}) \cdot (1 + \frac{v-1}{2v^2 - v}) = \frac{2v^3 - 2v^2 - v + 1}{12v^3 - 6v^2} \tag{25}$$

2) Assume now that player 1 won in the first stage (node E in Figure 1). Then if player 1 wins again in the second stage, he wins the entire tournament and his payoff is 1. However, if player 1 loses in the second

stage and player 2 wins in the third stage, player 1's expected payoff is $\frac{1}{3}$. By (14), if player 3 wins in the second stage, his expected payoff in the next stage is $v-\frac{1}{3}$.

Now, we have to analyze two different subcases of asymmetry, i.e., $1<v\leq 1.267$ and $v>1.267$. First, assume that $1<v\leq 1.267$ (node E1 in Figure 1). Then, by (14), (15), there is a unique mixed strategy equilibrium in which players 1 and 3 randomize on the interval $[0,\frac{3v-1}{3}]$ according to their effort cumulative distribution functions $F_i^2, i=1,3$ which are given by

$$\begin{aligned} 1\cdot F_3^2(x)+(\frac{1}{4v}\cdot\frac{1}{3})\cdot(1-F_3^2(x))-x &= \frac{-3v+4}{3} \\ (v-\frac{1}{3})\cdot F_1^2(x)-x &= 0 \end{aligned} \tag{26}$$

Then, player 1's winning probability against player 3 in the second stage (node E1 in Figure 1) is given by

$$p_{13}^{E1}=1-\frac{6v^2-2v}{12v-1} \tag{27}$$

and the expected total effort in the second stage (node E1 in Figure 1) is given by

$$TE^{E1}=(\frac{3v-1}{6})\cdot(1+\frac{12v^2-4v}{12v-1})=\frac{36v^3+12v^2-11v+1}{72v-6} \tag{28}$$

If, however, $v>1.267$ (node E2 in Figure 1), then, as previously, by (14), (15), there is a unique mixed strategy equilibrium in which players 1 and 3 randomize on the interval $[0,\frac{12v-1}{12v}]$ according to their effort cumulative distribution functions $F_i^2, i=1,3$ which are given by

$$\begin{aligned} 1\cdot F_3^2(x)+(\frac{1}{4v}\cdot\frac{1}{3})\cdot(1-F_3^2(x))-x &= \frac{1}{12v} \\ (v-\frac{1}{3})\cdot F_1^2(x)-x &= \frac{12v^2-16v+1}{12v} \end{aligned} \tag{29}$$

Then, player 1's winning probability against player 3 in the second stage (node E2 in Figure 1) is given by

$$p_{13}^{E2}=\frac{12v-1}{24v^2-8v} \tag{30}$$

and the expected total effort in the second stage (node E2 in Figure 1) is given by

$$TE^{E2}=(\frac{12v-1}{24v})\cdot(1+\frac{12v-1}{12v^2-4v})=\frac{144v^3+84v^2-20v+1}{288v^3-96v^2} \tag{31}$$

8.3 Stage 1 (player 1 vs. player 2)

If $1 < v \leq 1.09$ (node F1 in Figure 1), by (5), (8), (14), (17), (18), (26) and (27), there is a unique mixed strategy equilibrium in which players 1 and 2 randomize on the interval $[0, \frac{4-v}{12}]$ according to their effort cumulative distribution functions $F_i^1, i = 1, 2$ which are given by

$$\begin{aligned}(\frac{-3v+4}{3}) \cdot F_2^1(x) + (\frac{-11v+12}{12}) \cdot (1 - F_2^1(x)) - x &= \frac{-11v+12}{12} \\ (\frac{5v^2-21v+18}{3v}) \cdot F_1^1(x) - x &= \frac{21v^2-88v+72}{12v}\end{aligned} \tag{32}$$

Then, player 1's winning probability against player 2 in the first stage (node F1 in Figure 1) is given by

$$p_{12}^{F1} = \frac{v^2-4v}{-40v^2+168v-144} \tag{33}$$

and the expected total effort in the first stage (node F1 in Figure 1) is given by

$$TE^{F1} = (\frac{-v+4}{24}) \cdot (1 + \frac{v^2-4v}{-20v^2+84v-72}) = \frac{19v^3-156v^2+392v-288}{-480v^2+2016v-1728} \tag{34}$$

If $1.09 < v \leq 1.267$ (node F2 in Figure 1), by (5), (8), (14), (20), (21), (26) and (27), there is a unique mixed strategy equilibrium in which players 1 and 2 randomize on the interval $[0, \frac{-3v+4}{3}]$ according to their effort cumulative distribution functions $F_i^1, i = 1, 2$ which are given by

$$\begin{aligned}(\frac{-3v+4}{3}) \cdot F_2^1(x) - x &= 0 \\ (\frac{-v^2+3v}{72v-72}) \cdot F_1^1(x) - x &= \frac{71v^2-165v+96}{72v-72}\end{aligned} \tag{35}$$

Then, player 1's winning probability against player 2 in the first stage (node F2 in Figure 1) is given by

$$p_{12}^{F2} = \frac{36v^2-84v+48}{v^2-3v} \tag{36}$$

and the expected total effort in the first stage (node F2 in Figure 1) is given by

$$TE^{F2} = (\frac{-3v+4}{6}) \cdot (1 + \frac{72v^2-168v+96}{v^2-3v}) = \frac{-219v^3+805v^2-972v+384}{6v^2-18v} \tag{37}$$

If $1.267 < v \leq 1.596$ (node F3 in Figure 1), by (5), (8), (14), (20), (21), (29) and (30), there is a unique mixed strategy equilibrium in which players 1 and 2 randomize on the interval $[0, \frac{1}{12v}]$ according to their effort cumulative distribution functions $F_i^1, i = 1, 2$ which are given by

$$\begin{aligned} \frac{1}{12v} \cdot F_2^1(x) - x &= 0 \\ (\frac{-v^2 + 3v}{72v - 72}) \cdot F_1^1(x) - x &= \frac{v^3 - 3v^2 + 6v - 6}{-72v^2 + 72v} \end{aligned} \tag{38}$$

Then, player 1's winning probability against player 2 in the first stage (node F3 in Figure 1) is given by

$$p_{12}^{F3} = \frac{-3v + 3}{v^3 - 3v^2} \tag{39}$$

and the expected total effort in the first stage (node F3 in Figure 1) is given by

$$TE^{F3} = (\frac{1}{24v}) \cdot (1 + \frac{-6v + 6}{v^3 - 3v^2}) = \frac{v^3 - 3v^2 - 6v + 6}{24v^4 - 72v^3} \tag{40}$$

Similarly, if $1.596 < v \leq 2$ (node F4 in Figure 1), by (5), (8), (14), (20), (21), (29) and (30), there is a unique mixed strategy equilibrium in which players 1 and 2 randomize on the interval $[0, \frac{-v^2+3v}{72v-72}]$ according to their effort cumulative distribution functions $F_i^1, i = 1, 2$ which are given by

$$\begin{aligned} \frac{1}{12v} \cdot F_2^1(x) - x &= \frac{v^3 - 3v^2 + 6v - 6}{72v^2 - 72v} \\ (\frac{-v^2 + 3v}{72v - 72}) \cdot F_1^1(x) - x &= 0 \end{aligned} \tag{41}$$

Then, player 1's winning probability against player 2 in the first stage (node F4 in Figure 1) is given by

$$p_{12}^{F4} = 1 - \frac{v^3 - 3v^2}{-12v + 12} \tag{42}$$

and the expected total effort in the first stage (node F4 in Figure 1) is given by

$$TE^{F4} = (\frac{-v^2 + 3v}{144v - 144}) \cdot (1 + \frac{v^3 - 3v^2}{-6v + 6}) = \frac{v^5 - 6v^4 + 3v^3 + 24v^2 - 18v}{864v^2 - 1728v + 864} \tag{43}$$

Finally, if $v > 2$ (node F5 in Figure 1), by (5), (11), (14), (23), (24), (29) and (30), there is a unique mixed strategy equilibrium in which players 1 and 2 randomize on the interval $[0, \frac{v-1}{12v^2-6v}]$ according to their effort cumulative distribution functions $F_i^1, i = 1, 2$ which are given by

$$\begin{aligned} \frac{1}{12v} \cdot F_2^1(x) - x &= \frac{1}{24v^2 - 12v} \\ (\frac{v - 1}{12v^2 - 6v}) \cdot F_1^1(x) - x &= 0 \end{aligned} \tag{44}$$

Then, player 1's winning probability against player 2 in the first stage (node F5 in Figure 1) is given by

$$p_{12}^{F5} = 1 - \frac{v - 1}{2v - 1} \tag{45}$$

And, the expected total effort in the first stage (node F5 in Figure 1) is given by

$$TE^{F5} = \left(\frac{v-1}{24v^2-12v}\right) \cdot \left(1 + \frac{2v-2}{2v-1}\right) = \frac{4v^2-7v+3}{48v^3-48v^2+12v} \tag{46}$$

9 Appendix B: The weak players are matched in the second stage of the round-robin tournament

We analyze here the case where the weak players (players 1 and 2) are matched in the second stage. The possible paths of this tournament are described by Figure 2.

9.1 Stage 3 (player 2 vs. player 3)

We have the following three scenarios:

1) Assume first that player 3 won the game in the first stage and player 2 won the game in the second stage (node A in Figure 2). Therefore, each player wins once before the last stage such that the winner of the third stage wins the entire tournament. In that case player 2's payoff is 1, and player 3's is v. The loser's payoff in that stage is zero. In the unique mixed strategy equilibrium, players 2 and 3 randomize on the interval $[0, 1]$ according to their effort cumulative distribution functions $F_i^3, i = 2, 3$ which are given by

$$\begin{aligned} 1 \cdot F_3^3(x) - x &= 0 \\ v \cdot F_2^3(x) - x &= v - 1 \end{aligned} \tag{47}$$

Then, player 2's winning probability against player 3 in the third stage (node A in Figure 2) is given by

$$p_{23}^A = \frac{1}{2v} \tag{48}$$

and the expected total effort in the third stage (node A in Figure 2) is given by

$$TE^A = \frac{1}{2} \cdot \left(1 + \frac{1}{v}\right) = \frac{v+1}{2v} \tag{49}$$

2) Assume now that player 3 won the game in the first stage and player 2 lost the game in the second stage (node B in Figure 2). Thus, if player 3 wins in the third stage, he wins the entire tournament. Then

his payoff is v, whereas player 2's payoff is zero. But if player 2 wins in this stage, then every player wins once and a draw will determine the winner of the tournament. Then, players 1 and 2's payoff is equal to $\frac{1}{3}$ and player 3's is $\frac{v}{3}$. Thus, there is a unique mixed strategy equilibrium in which players 2 and 3 randomize on the interval $[0, \frac{1}{3}]$ according to their effort cumulative distribution functions $F_i^3, i = 2, 3$ which are given by

$$\begin{aligned} \frac{1}{3} \cdot F_3^3(x) - x &= 0 \\ v \cdot F_2^3(x) + \frac{v}{3} \cdot (1 - F_2^3(x)) - x &= v - \frac{1}{3} \end{aligned} \tag{50}$$

Then, player 2's winning probability against player 3 in the third stage (node B in Figure 2) is given by

$$p_{23}^B = \frac{1}{4v} \tag{51}$$

and the expected total effort in the third stage (node B in Figure 2) is given by

$$TE^B = \frac{1}{6} \cdot (1 + \frac{1}{2v}) = \frac{2v+1}{12v} \tag{52}$$

3) Assume now that player 3 lost the game in the first stage and player 2 won the game in the second stage (node C in Figure 2). Thus, if player 2 also wins in the third stage he wins the entire tournament. Then, his payoff is 1, whereas player 3's payoff is zero. But if player 3 wins in this stage, then every player wins once and a draw will determine the winner of the tournament. Then, players 1 and 2's payoff is equal to $\frac{1}{3}$ and player 3's payoff is equal to $\frac{v}{3}$. In that case we have to consider two different subcases of asymmetry, i.e., $1 < v \leq 2$ and $v > 2$. If $1 < v \leq 2$ (node C1 in Figure 2), there is a unique mixed strategy equilibrium in which players 2 and 3 randomize on the interval $[0, \frac{v}{3}]$ according to their effort cumulative distribution functions $F_i^3, i = 2, 3$ which are given by

$$\begin{aligned} 1 \cdot F_3^3(x) + \frac{1}{3} \cdot (1 - F_3^3(x)) - x &= 1 - \frac{v}{3} \\ \frac{v}{3} \cdot F_2^3(x) - x &= 0 \end{aligned} \tag{53}$$

Then, player 2's winning probability against player 3 in the third stage (node C1 in Figure 2) is given by

$$p_{23}^{C1} = 1 - \frac{v}{4} \tag{54}$$

and the expected total effort in the third stage (node C1 in Figure 2) is given by

$$TE^{C1} = \frac{v}{6} \cdot (1 + \frac{v}{2}) = \frac{v^2 + 2v}{12} \tag{55}$$

If, however, $v > 2$ (node C2 in Figure 2), there is a unique mixed strategy equilibrium in which players 2 and 3 randomize on the interval $[0, \frac{2}{3}]$ according to their effort cumulative distribution functions $F_i^3, i = 2, 3$ which are given by

$$\begin{aligned} 1 \cdot F_3^3(x) + \frac{1}{3} \cdot (1 - F_3^3(x)) - x &= \frac{1}{3} \\ \frac{v}{3} \cdot F_2^3(x) - x &= \frac{v-2}{3} \end{aligned} \tag{56}$$

Then, player 2's winning probability against player 3 in the third stage (node C2 in Figure 2) is given by

$$p_{23}^{C2} = \frac{1}{v} \tag{57}$$

and the expected total effort in the third stage (node C2 in Figure 2) is given by

$$TE^{C2} = \frac{1}{3} \cdot (1 + \frac{2}{v}) = \frac{v+2}{3v} \tag{58}$$

9.2 Stage 2 (player 1 vs. player 2)

We have here two possible scenarios:

1) Assume that player 1 lost in the first stage (node D in Figure 2). Then by (47), even if player 2 wins in the second stage, his expected payoff in the next stage is zero. Therefore he has no incentive to exert a positive effort and we actually do not have an equilibrium. However, as we already mentioned, in order to solve this problem, we can assume that each player obtains a payment of $k > 0$, if he wins a single game. Then we consider the limit behavior as $k \to 0$. This assumption does not affect the players' behavior, but ensures that equilibrium exists. Then, player 1's winning probability against player 2 in the second stage (node D in Figure 2) is given by

$$p_{12}^{D} = 1 \tag{59}$$

and the expected total effort in the second stage (node D in Figure 2) is given by

$$TE^{D} = 0 \tag{60}$$

2) However, if player 1 wins in the first stage (node E of Figure 2), then if player 1 wins again in the second stage, he wins the entire tournament and his payoff is 1. If, on the other hand, player 1 loses in the second stage and player 3 wins in the third stage, his expected payoff is $\frac{1}{3}$ (node C in Figure 2). Now, we have to analyze two different subcases of asymmetry, i.e. $1 < v \leq 2$ and $v > 2$. First, assume that $1 < v \leq 2$ (node E1 in Figure 2). Then by (53), if player 2 wins in the second stage, his expected payoff in the next stage is $1 - \frac{v}{3}$. Thus, by (53) and (54), there is a unique mixed strategy equilibrium in which players 1 and 2 randomize on the interval $[0, \frac{3-v}{3}]$ according to their effort cumulative distribution functions $F_i^2, i = 1, 2$ which are given by

$$\begin{aligned} 1 \cdot F_2^2(x) + (\frac{v}{4} \cdot \frac{1}{3}) \cdot (1 - F_2^2(x)) - x &= \frac{v}{3} \\ (\frac{3-v}{3}) \cdot F_1^2(x) - x &= 0 \end{aligned} \tag{61}$$

Then, player 1's winning probability against player 2 in the second stage (node E1 in Figure 2) is given by

$$p_{12}^{E1} = 1 - \frac{2v-6}{v-12} \tag{62}$$

and the expected total effort in the second stage (node E1 in Figure 2) is given by

$$TE^{E1} = (\frac{-v+3}{6}) \cdot (1 + \frac{4v-12}{v-12}) = \frac{-5v^2 + 39v - 72}{6v - 72} \tag{63}$$

If, however, $v > 2$ (node E2 in Figure 2), then by (56), if player 2 wins in the second stage, his expected payoff in the next stage is $\frac{1}{3}$. Thus, by (56) and (57), there is a unique mixed strategy equilibrium in which players 1 and 2 randomize on the interval $[0, \frac{1}{3}]$ according to their effort cumulative distribution functions $F_i^2, i = 1, 2$ which are given by

$$\begin{aligned} 1 \cdot F_2^2(x) + ((1 - \frac{1}{v}) \cdot \frac{1}{3}) \cdot (1 - F_2^2(x)) - x &= \frac{2}{3} \\ \frac{1}{3} \cdot F_1^2(x) - x &= 0 \end{aligned} \tag{64}$$

Then, player 1's winning probability against player 2 in the second stage (node E2 in Figure 2) is given by

$$p_{12}^{E2} = 1 - \frac{v}{4v+2} \tag{65}$$

and the expected total effort in the second stage (node E2 in Figure 2) is given by

$$TE^{E2} = \frac{1}{6} \cdot (1 + \frac{v}{2v+1}) = \frac{3v+1}{12v+6} \tag{66}$$

9.3 Stage 1 (player 1 vs. player 3)

If $1 < v \leq 2$ (node F1 in Figure 2), by (50), (53), (59), (61) and (62), there is a unique mixed strategy equilibrium in which players 1 and 3 randomize on the interval $[0, \frac{4v^2-1}{12v}]$ according to their effort cumulative distribution functions $F_i^1, i = 1, 3$ which are given by

$$\begin{aligned} \frac{v}{3} \cdot F_3^1(x) + (\frac{1}{3} \cdot \frac{1}{4v}) \cdot (1 - F_3^1(x)) - x &= \frac{1}{12v} \\ (\frac{3v-1}{3}) \cdot F_1^1(x) - x &= \frac{8v^2 - 4v + 1}{12v} \end{aligned} \tag{67}$$

Then, player 1's winning probability against player 3 in the first stage (node F1 in Figure 2) is given by

$$p_{13}^{F1} = \frac{4v^2 - 1}{24v^2 - 8v} \tag{68}$$

and the expected total effort in the first stage (node F1 in Figure 2) is given by

$$TE^{F1} = (\frac{4v^2-1}{24v}) \cdot (1 + \frac{4v^2-1}{12v^2 - 4v}) = \frac{64v^4 - 16v^3 - 20v^2 + 4v + 1}{288v^3 - 96v^2} \tag{69}$$

If, on the other hand, $v > 2$ (node F2 in Figure 2), by (50), (56), (59), (64) and (65), there is a unique mixed strategy equilibrium in which players 1 and 3 randomize on the interval $[0, \frac{8v-1}{12v}]$ according to their effort cumulative distribution functions $F_i^1, i = 1, 3$ which are given by

$$\begin{aligned} \frac{2}{3} \cdot F_3^1(x) + (\frac{1}{3} \cdot \frac{1}{4v}) \cdot (1 - F_3^1(x)) - x &= \frac{1}{12v} \\ (\frac{3v-1}{3}) \cdot F_1^1(x) + (\frac{v^2-2v}{12v+6}) \cdot (1 - F_1^1(x)) - x &= \frac{12v^2 - 12v + 1}{12v} \end{aligned} \tag{70}$$

Then, player 1's winning probability against player 3 in the first stage (node F2 in Figure 2) is given by

$$p_{13}^{F2} = \frac{16v^2 + 6v - 1}{44v^3 + 16v^2 - 8v} \tag{71}$$

and the expected total effort in the first stage (node F2 in Figure 2) is given by

$$TE^{F2} = (\frac{8v-1}{24v}) \cdot (1 + \frac{16v^2 + 6v - 1}{22v^3 + 8v^2 - 4v}) = \frac{176v^4 + 170v^3 - 8v^2 - 10v + 1}{528v^4 + 192v^3 - 96v^2} \tag{72}$$

10 Appendix C: The weak players are matched in the third stage of the round-robin tournament

We now analyze the case where the weak players (players 1 and 2) are matched in the last stage. The possible paths of this tournament are described by Figure 3.

10.1 Stage 3 (player 1 vs. player 2)

As previously, we have the following three scenarios:

1) Assume first that player 1 won the game in the first stage and player 2 won the game in the second stage (node A in Figure 3). Therefore, each player won once before the last stage, implying that the winner of the third stage wins the entire contest. In that case, the winner's payoff would be 1 and the loser's payoff would be zero. Thus, there is a unique mixed strategy equilibrium in which players 1 and 2 randomize on the interval $[0,1]$ according to their effort cumulative distribution functions $F_i^3, i = 1,2$ which are given by

$$\begin{aligned} 1 \cdot F_2^3(x) - x &= 0 \\ 1 \cdot F_1^3(x) - x &= 0 \end{aligned} \tag{73}$$

Then, player 1's winning probability against player 2 in the third stage (node A in Figure 3) is given by

$$p_{12}^A = \frac{1}{2} \tag{74}$$

and the expected total effort in the third stage (node A in Figure 3) is given by

$$TE^A = \frac{1}{2} \cdot (1+1) = 1 \tag{75}$$

2) Assume now that player 1 won in the first stage and player 2 lost in the second stage (node B in Figure 3). Thus, if player 1 also wins in the third stage he wins the tournament. Then, his payoff is 1, whereas player 2's payoff is zero. But, if player 2 wins in this stage, then every player wins once and a draw will determine the winner of the tournament. In that case, players 1 and 2's payoff is equal to $\frac{1}{3}$ and player 3's payoff is equal to $\frac{v}{3}$. Thus, there is a unique mixed strategy equilibrium in which players 1 and 2 randomize

on the interval $[0, \frac{1}{3}]$ according to their effort cumulative distribution functions $F_i^3, i = 1, 2$ which are given by

$$\begin{aligned} 1 \cdot F_2^3(x) + \frac{1}{3} \cdot (1 - F_2^3(x)) - x &= \frac{2}{3} \\ \frac{1}{3} \cdot F_1^3(x) - x &= 0 \end{aligned} \tag{76}$$

Then, player 1's winning probability against player 2 in the third stage (node B in Figure 3) is given by

$$p_{12}^B = \frac{3}{4} \tag{77}$$

and the expected total effort in the third stage (node B in Figure 3) is given by

$$TE^B = \frac{1}{6} \cdot (1 + \frac{1}{2}) = \frac{1}{4} \tag{78}$$

3) Assume now that player 1 lost the game in the first stage and player 2 won in the second stage (node C in Figure 3). Thus, if player 2 wins in the third stage he wins the entire tournament. Then, his payoff is 1, whereas player 1's payoff is zero. But, if player 1 wins in this stage, then every player wins once and a draw will determine the winner of the tournament. In that case, players 1 and 2's payoff is equal to $\frac{1}{3}$ and player 3's payoff is equal to $\frac{v}{3}$. Thus, there is a unique mixed strategy equilibrium in which players 1 and 2 randomize on the interval $[0, \frac{1}{3}]$ according to their effort cumulative distribution functions $F_i^3, i = 1, 2$ which are given by

$$\begin{aligned} \frac{1}{3} \cdot F_2^3(x) - x &= 0 \\ 1 \cdot F_1^3(x) + \frac{1}{3} \cdot (1 - F_1^3(x)) - x &= \frac{2}{3} \end{aligned} \tag{79}$$

Then, player 1's winning probability against player 2 in the third stage (node C in Figure 3) is given by

$$p_{12}^C = \frac{1}{4} \tag{80}$$

and the expected total effort in the third stage (node C in Figure 3) is given by

$$TE^C = \frac{1}{6} \cdot (1 + \frac{1}{2}) = \frac{1}{4} \tag{81}$$

10.2 Stage 2 (player 2 vs. player 3)

We have here two possible scenarios:

1) Assume that player 3 lost in the first stage (node D in Figure 3). Then, by (73), even if player 2 wins in the second stage, his expected payoff in the next stage is zero. Then, player 2's winning probability against player 3 in the second stage (node D in Figure 3) is given by

$$p_{23}^{D} = 0 \tag{82}$$

and the expected total effort in the second stage (node D in Figure 3) is given by

$$TE^{D} = 0 \tag{83}$$

2) However, if player 3 won in the first stage (node E in Figure 3), then if player 3 wins again in the second stage, he wins the entire tournament and his payoff is v. However, if he loses in this stage and player 1 wins in the third stage, player 3's expected payoff is $\frac{v}{3}$. Then, by (79), if player 2 wins in the second stage, his expected payoff in the next stage is $\frac{2}{3}$. Thus, by (79) and (80), there is a unique mixed strategy equilibrium in which players 2 and 3 randomize on the interval $[0, \frac{2}{3}]$ according to their effort cumulative distribution functions $F_i^2, i = 2, 3$ which are given by

$$\begin{aligned} \frac{2}{3} \cdot F_3^2(x) - x &= 0 \qquad (84) \\ v \cdot F_2^2(x) + (\frac{1}{4} \cdot \frac{v}{3}) \cdot (1 - F_2^2(x)) - x &= v - \frac{2}{3} \end{aligned}$$

Then, player 2's winning probability against player 3 in the second stage (node E in Figure 3) is given by

$$p_{23}^{E} = \frac{4}{11v} \tag{85}$$

and the expected total effort in the second stage (node E in Figure 3) is given by

$$TE^{E} = \frac{1}{3} \cdot (1 + \frac{8}{11v}) = \frac{11v + 8}{33v} \tag{86}$$

10.3 Stage 1 (player 1 vs. player 3)

If $1 < v \leq 1.45$ (node F1 in Figure 3), by (76), (79), (82), (84) and (85), there is a unique mixed strategy equilibrium in which players 1 and 3 randomize on the interval $[0, \frac{11v-8}{12}]$ according to their effort cumulative distribution functions $F_i^1, i = 1, 3$ which are given by

$$\begin{aligned} \frac{2}{3} \cdot F_3^1(x) - x &= \frac{-11v+16}{12} \\ (\frac{3v-2}{3}) \cdot F_1^1(x) + (\frac{1}{4} \cdot \frac{v}{3}) \cdot (1 - F_1^1(x)) - x &= \frac{v}{12} \end{aligned} \quad (87)$$

Then, player 1's winning probability against player 3 in the first stage (node F1 in Figure 3) is given by

$$p_{13}^{F1} = 1 - \frac{11v-8}{16} \quad (88)$$

and the expected total effort in the first stage (node F1 in Figure 3) is given by

$$TE^{F1} = (\frac{11v-8}{24}) \cdot (1 + \frac{11v-8}{8}) = \frac{121v^2 - 88v}{192} \quad (89)$$

Finally, if $v > 1.45$ (node F2 in Figure 3), by (76), (79), (82), (84) and (85), there is a unique mixed strategy equilibrium in which players 1 and 3 randomize on the interval $[0, \frac{2}{3}]$ according to their effort cumulative distribution functions $F_i^1, i = 1, 3$ which are given by

$$\begin{aligned} \frac{2}{3} \cdot F_3^1(x) - x &= 0 \\ (\frac{3v-2}{3}) \cdot F_1^1(x) + (\frac{1}{4} \cdot \frac{v}{3}) \cdot (1 - F_1^1(x)) - x &= \frac{3v-4}{3} \end{aligned} \quad (90)$$

Then, player 1's winning probability against player 3 in the first stage (node F2 in Figure 3) is given by

$$p_{13}^{F2} = \frac{4}{11v-8} \quad (91)$$

and the expected total effort in the first stage (node F2 in Figure 3) is given by

$$TE^{F2} = \frac{1}{3} \cdot (1 + \frac{8}{11v-8}) = \frac{11v}{33v-24} \quad (92)$$

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11 Appendix D - Online Appendix

11.1 Proof of Proposition 1

Case A: When the weak players are matched in the first stage, the players' expected payoffs are obtained by the analysis in Appendix A. By this analysis we obtain that

1) If $1 < v \leq 1.09$, player 1's expected payoff is $\frac{-11v+12}{12}$, player 2's is $\frac{21v^2-88v+72}{12v}$ and player 3's is zero.

2) If $1.09 < v \leq 1.267$, player 1's expected payoff is zero, player 2's is $\frac{71v^2-165v+96}{72v-72}$ and player 3's is $\frac{-385v^3+1311v^2-1500v+576}{12v^2-36v}$.

3) If $1.267 < v \leq 1.596$, player 1's expected payoff is zero, player 2's is $\frac{v^3-3v^2+6v-6}{-72v^2+72v}$ and player 3's is $\frac{11v^5-45v^4+33v^3+15v^2-15v+3}{12v^4-36v^3}$.

4) If $1.596 < v \leq 2$, player 1's expected payoff is $\frac{v^3-3v^2+6v-6}{72v^2-72v}$, player 2's is zero, and player 3's is $\frac{v^5-7v^4+157v^3-339v^2+204v-12}{144v^2-144v}$.

5) if $v > 2$, player 1's expected payoff is $\frac{1}{24v^2-12v}$, player 2's is zero, and player 3's is $\frac{24v^3-44v^2+21v-4}{24v^2-12v}$.

We can see that when the asymmetry is weak, i.e., $1 < v \leq 1.09$, the dominant player (player 3) has an expected payoff of zero which is smaller than the expected payoffs of the other players, while when the asymmetry is strong, i.e., $v > 2$, the dominant player's expected payoff is higher than the other players since $24v^3 - 44v^2 + 21v - 5 > 0$ for every $v > 2$. Thus, if the asymmetry between the players is relatively low, the expected payoff of the dominant player is lower than the expected payoffs of the weak players.

Case B: When the weak players are matched in the second stage, the players' expected payoffs are obtained by the analysis in Appendix B. By this analysis we obtain that

1) if $1 < v \leq 2$, player 1's expected payoff is $\frac{1}{12v}$, player 2's is zero, and player 3's is $\frac{8v^2-4v+1}{12v}$.

2) if $v > 2$, player 1's expected payoff is $\frac{1}{12v}$, player 2's is zero, and player 3's is $\frac{12v^2-12v+1}{12v}$.

We can see that when $1 < v \leq 2$ the dominant player's expected payoff is higher than the other players since $8v^2 - 4v > 0$. When $v > 2$, since $12v^2 - 12v > 0$ the dominant player has the highest expected payoff. Thus, independent of whether asymmetry is weak or strong, the expected payoff of the dominant player is always higher than the expected payoffs of the weak players.

Case C: When the weak players are matched in the third stage, the players' expected payoffs are obtained by the analysis in Appendix C. By this analysis we obtain that

1) if $1 < v \leq 1.45$, player 1's expected payoff is $\frac{-11v+16}{12}$, player 2's is zero, and player 3's is $\frac{v}{12}$.

2) if $v > 1.45$, player 1's expected payoff is zero, player 2's is zero, and player 3's is $\frac{3v-4}{3}$.

We can see that when $1 < v \leq 1.33$ since $-12v + 16 > 0$, the dominant player does not have the highest expected payoff. On the other hand, when $v > 1.33$, since $-12v + 16 < 0$ and $3v - 4 > 0$, the dominant player has the highest expected payoff. Thus, if the asymmetry between the players is relatively low, the expected payoff of the dominant player is lower than the expected payoffs of the weak players. Q.E.D.

11.2 The dominant player's winning probability

In the following we calculate the dominant player's probability of winning in the round-robin tournaments for all the three possible allocations of players.

11.2.1 Case A: The weak players are matched in the first stage (stage 1: 1 vs. 2, stage 2: 1 vs. 3, stage 3: 2 vs. 3)

In this case if $1 < v \leq 1.09$, by (6), (9), (15), (18), (27) and (33) the dominant player's (player 3) probability to win the round-robin tournament is

$$\begin{aligned} p_3^{RR} &= p_{21}^{F1} \cdot p_{31}^{D1} \cdot p_{32}^{A} + p_{21}^{F1} \cdot p_{13}^{D1} \cdot p_{32}^{B1} \cdot \frac{1}{3} + p_{12}^{F1} \cdot p_{31}^{E1} \cdot p_{32}^{C} + p_{12}^{F1} \cdot p_{31}^{E1} \cdot p_{23}^{C} \cdot \frac{1}{3} \\ &= (1 - \frac{v^2 - 4v}{-40v^2 + 168v - 144}) \cdot (\frac{6v-6}{v}) \cdot (1 - \frac{1}{2v}) \\ &\quad + \frac{1}{3} \cdot (1 - \frac{v^2 - 4v}{-40v^2 + 168v - 144}) \cdot (1 - \frac{6v-6}{v}) \cdot \frac{v}{4} \\ &\quad + (\frac{v^2 - 4v}{-40v^2 + 168v - 144}) \cdot (\frac{6v^2 - 2v}{12v - 1}) \cdot (1 - \frac{1}{4v}) \\ &\quad + \frac{1}{3} \cdot (\frac{v^2 - 4v}{-40v^2 + 168v - 144}) \cdot (\frac{6v^2 - 2v}{12v - 1}) \cdot \frac{1}{4v} \end{aligned}$$

If $1.09 < v \leq 1.267$, by (6), (9), (15), (21), (27) and (36) the dominant player's probability to win the round-robin tournament is

$$\begin{aligned}
p_3^{RR} &= p_{21}^{F2}\cdot p_{31}^{D2}\cdot p_{32}^{A} + p_{21}^{F2}\cdot p_{13}^{D2}\cdot p_{32}^{B1}\cdot\frac{1}{3} + p_{12}^{F2}\cdot p_{31}^{E1}\cdot p_{32}^{C} + p_{12}^{F2}\cdot p_{31}^{E1}\cdot p_{23}^{C}\cdot\frac{1}{3}\\
&= (1-\frac{36v^2-84v+48}{v^2-3v})\cdot(1-\frac{v}{24v-24})\cdot(1-\frac{1}{2v})\\
&\quad+\frac{1}{3}\cdot(1-\frac{36v^2-84v+48}{v^2-3v})\cdot(\frac{v}{24v-24})\cdot\frac{v}{4}\\
&\quad+(\frac{36v^2-84v+48}{v^2-3v})\cdot(\frac{6v^2-2v}{12v-1})\cdot(1-\frac{1}{4v})\\
&\quad+\frac{1}{3}\cdot(\frac{36v^2-84v+48}{v^2-3v})\cdot(\frac{6v^2-2v}{12v-1})\cdot\frac{1}{4v}
\end{aligned}$$

If $1.267 < v \leq 1.596$, by (6), (9), (15), (21), (30) and (39) the dominant player's probability to win the round-robin tournament is

$$\begin{aligned}
p_3^{RR} &= p_{21}^{F3}\cdot p_{31}^{D2}\cdot p_{32}^{A} + p_{21}^{F3}\cdot p_{13}^{D2}\cdot p_{32}^{B1}\cdot\frac{1}{3} + p_{12}^{F3}\cdot p_{31}^{E2}\cdot p_{32}^{C} + p_{12}^{F3}\cdot p_{31}^{E2}\cdot p_{23}^{C}\cdot\frac{1}{3}\\
&= (1-\frac{-3v+3}{v^3-3v^2})\cdot(1-\frac{v}{24v-24})\cdot(1-\frac{1}{2v})\\
&\quad+\frac{1}{3}\cdot(1-\frac{-3v+3}{v^3-3v^2})\cdot(\frac{v}{24v-24})\cdot\frac{v}{4}\\
&\quad+(\frac{-3v+3}{v^3-3v^2})\cdot(1-\frac{12v-1}{24v^2-8v})\cdot(1-\frac{1}{4v})\\
&\quad+\frac{1}{3}\cdot(\frac{-3v+3}{v^3-3v^2})\cdot(1-\frac{12v-1}{24v^2-8v})\cdot\frac{1}{4v}
\end{aligned}$$

If $1.596 < v \leq 2$, by (6), (9), (15), (21), (30) and (42) the dominant player's probability to win the round-robin tournament is

$$\begin{aligned}
p_3^{RR} &= p_{21}^{F4}\cdot p_{31}^{D2}\cdot p_{32}^{A} + p_{21}^{F4}\cdot p_{13}^{D2}\cdot p_{32}^{B1}\cdot\frac{1}{3} + p_{12}^{F4}\cdot p_{31}^{E2}\cdot p_{32}^{C} + p_{12}^{F4}\cdot p_{31}^{E2}\cdot p_{23}^{C}\cdot\frac{1}{3}\\
&= (\frac{v^3-3v^2}{-12v+12})\cdot(1-\frac{v}{24v-24})\cdot(1-\frac{1}{2v})\\
&\quad+\frac{1}{3}\cdot(\frac{v^3-3v^2}{-12v+12})\cdot(\frac{v}{24v-24})\cdot\frac{v}{4}\\
&\quad+(1-\frac{v^3-3v^2}{-12v+12})\cdot(1-\frac{12v-1}{24v^2-8v})\cdot(1-\frac{1}{4v})\\
&\quad+\frac{1}{3}\cdot(1-\frac{v^3-3v^2}{-12v+12})\cdot(1-\frac{12v-1}{24v^2-8v})\cdot\frac{1}{4v}
\end{aligned}$$

And if $v > 2$, by (6), (12), (15), (24), (30) and (45) the stronger player's probability to win the round-robin tournament is

$$\begin{aligned}
p_3^{RR} &= p_{21}^{F5}\cdot p_{31}^{D3}\cdot p_{32}^{A}+p_{21}^{F5}\cdot p_{13}^{D3}\cdot p_{32}^{B2}\cdot\frac{1}{3}+p_{12}^{F5}\cdot p_{31}^{E2}\cdot p_{32}^{C}+p_{12}^{F5}\cdot p_{31}^{E2}\cdot p_{23}^{C}\cdot\frac{1}{3}\\
&= (\frac{v-1}{2v-1})\cdot(1-\frac{v-1}{4v^2-2v})\cdot(1-\frac{1}{2v})\\
&\quad+\frac{1}{3}\cdot(\frac{v-1}{2v-1})\cdot(\frac{v-1}{4v^2-2v})\cdot(1-\frac{1}{v})\\
&\quad+(1-\frac{v-1}{2v-1})\cdot(1-\frac{12v-1}{24v^2-8v})\cdot(1-\frac{1}{4v})\\
&\quad+\frac{1}{3}\cdot(1-\frac{v-1}{2v-1})\cdot(1-\frac{12v-1}{24v^2-8v})\cdot\frac{1}{4v}
\end{aligned}$$

11.2.2 Case B: The weak players are matched in the second stage (stage 1: 1 vs. 3, stage 2: 1 vs. 2, stage 3: 2 vs. 3)

In this case if $1 < v \leq 2$, by (48), (51), (54), (59), (62) and (68) the dominant player's probability to win the round-robin tournament is

$$\begin{aligned}
p_3^{RR} &= p_{31}^{F1}\cdot p_{12}^{D}\cdot p_{32}^{B}+p_{31}^{F1}\cdot p_{12}^{D}\cdot p_{23}^{B}\cdot\frac{1}{3}+p_{13}^{F1}\cdot p_{21}^{E1}\cdot p_{32}^{C1}\cdot\frac{1}{3}\\
&= (1-\frac{4v^2-1}{24v^2-8v})\cdot(1-\frac{1}{4v})\\
&\quad+\frac{1}{3}\cdot(1-\frac{4v^2-1}{24v^2-8v})\cdot\frac{1}{4v}\\
&\quad+\frac{1}{3}\cdot(\frac{4v^2-1}{24v^2-8v})\cdot(\frac{2v-6}{v-12})\cdot\frac{v}{4}
\end{aligned}$$

And if $v > 2$, by (48), (51), (57), (59), (65) and (71) the dominant player's probability to win the round-robin tournament is

$$\begin{aligned}
p_3^{RR} &= p_{31}^{F2}\cdot p_{12}^{D}\cdot p_{32}^{B}+p_{31}^{F2}\cdot p_{12}^{D}\cdot p_{23}^{B}\cdot\frac{1}{3}+p_{13}^{F2}\cdot p_{21}^{E2}\cdot p_{32}^{C2}\cdot\frac{1}{3}\\
&= (1-\frac{16v^2+6v-1}{44v^3+16v^2-8v})\cdot(1-\frac{1}{4v})\\
&\quad+\frac{1}{3}\cdot(1-\frac{16v^2+6v-1}{44v^3+16v^2-8v})\cdot\frac{1}{4v}\\
&\quad+\frac{1}{3}\cdot(\frac{16v^2+6v-1}{44v^3+16v^2-8v})\cdot(\frac{v}{4v+2})\cdot(1-\frac{1}{v})
\end{aligned}$$

11.2.3 Case C: The weak players are matched in the third stage (stage 1: 1 vs. 3, stage 2: 2 vs. 3, stage 3: 1 vs. 2)

In this case if $1 < v \leq 1.45$, by (74), (77), (80), (82), (85) and (88) the dominant player's probability to win the round-robin tournament is

$$\begin{aligned}
p_3^{RR} &= p_{31}^{F1} \cdot p_{32}^{E} + p_{31}^{F1} \cdot p_{23}^{E} \cdot p_{12}^{C} \cdot \frac{1}{3} + p_{13}^{F1} \cdot p_{32}^{D} \cdot p_{21}^{B} \cdot \frac{1}{3} \\
&= (\frac{11v-8}{16}) \cdot (1 - \frac{4}{11v}) \\
&\quad + \frac{1}{3} \cdot (\frac{11v-8}{16}) \cdot \frac{4}{11v} \cdot \frac{1}{4} \\
&\quad + \frac{1}{3} \cdot (1 - \frac{11v-8}{16}) \cdot \frac{1}{4}
\end{aligned}$$

And, if $v > 1.45$, by (74), (77), (80), (82), (85) and (91) the dominant player's probability to win the round-robin tournament is

$$\begin{aligned}
p_3^{RR} &= p_{31}^{F2} \cdot p_{32}^{E} + p_{31}^{F2} \cdot p_{23}^{E} \cdot p_{12}^{C} \cdot \frac{1}{3} + p_{13}^{F2} \cdot p_{32}^{D} \cdot p_{21}^{B} \cdot \frac{1}{3} \\
&= (1 - \frac{4}{11v-8}) \cdot (1 - \frac{4}{11v}) \\
&\quad + \frac{1}{3} \cdot (1 - \frac{4}{11v-8}) \cdot \frac{4}{11v} \cdot \frac{1}{4} \\
&\quad + \frac{1}{3} \cdot (\frac{4}{11v-8}) \cdot \frac{1}{4}
\end{aligned}$$

11.3 Total effort

Below, we analyze the expected total effort in the round-robin tournament for the three possible allocations of players.

11.3.1 Case A: The weak players are matched in the first stage (stage 1: 1 vs. 2, stage 2: 1 vs. 3, stage 3: 2 vs. 3)

In this case if $1 < v \leq 1.09$, by (7), (10), (16), (18), (19), (27), (28), (33) and (34) the players' expected total effort is

$$\begin{aligned}
TE^{RR} &= TE^{F1} + p_{12}^{F1} \cdot TE^{E1} + p_{21}^{F1} \cdot TE^{D1} + p_{12}^{F1} \cdot p_{31}^{E1} \cdot TE^{C} + p_{21}^{F1} \cdot p_{13}^{D1} \cdot TE^{B1} + p_{21}^{F1} \cdot p_{31}^{D1} \cdot TE^{A} \\
&= (\frac{19v^3 - 156v^2 + 392v - 288}{-480v^2 + 2016v - 1728}) + (\frac{v^2 - 4v}{-40v^2 + 168v - 144}) \cdot (\frac{36v^3 + 12v^2 - 11v + 1}{72v - 6}) \\
&\quad + (1 - \frac{v^2 - 4v}{-40v^2 + 168v - 144}) \cdot (\frac{13v^2 - 25v + 12}{2v}) + (\frac{v^2 - 4v}{-40v^2 + 168v - 144}) \cdot (\frac{6v^2 - 2v}{12v - 1}) \cdot (\frac{2v + 1}{12v}) \\
&\quad + (1 - \frac{v^2 - 4v}{-40v^2 + 168v - 144}) \cdot (1 - \frac{6v - 6}{v}) \cdot (\frac{v^2 + 2v}{12}) + (1 - \frac{v^2 - 4v}{-40v^2 + 168v - 144}) \cdot (\frac{6v - 6}{v}) \cdot (\frac{v + 1}{2v})
\end{aligned}$$

If $1.09 < v \leq 1.267$, by (7), (10), (16), (21), (22), (27), (28), (36) and (37) the players' expected total effort is

$$\begin{aligned}
TE^{RR} &= TE^{F2} + p_{12}^{F2} \cdot TE^{E1} + p_{21}^{F2} \cdot TE^{D2} + p_{12}^{F2} \cdot p_{31}^{E1} \cdot TE^{C} + p_{21}^{F2} \cdot p_{13}^{D2} \cdot TE^{B1} + p_{21}^{F2} \cdot p_{31}^{D2} \cdot TE^{A} \\
&= (\frac{-219v^3 + 805v^2 - 972v + 384}{6v^2 - 18v}) + (\frac{36v^2 - 84v + 48}{v^2 - 3v}) \cdot (\frac{36v^3 + 12v^2 - 11v + 1}{72v - 6}) \\
&\quad + (1 - \frac{36v^2 - 84v + 48}{v^2 - 3v}) \cdot (\frac{13v^2 - 12v}{288v - 288}) + (\frac{36v^2 - 84v + 48}{v^2 - 3v}) \cdot (\frac{6v^2 - 2v}{12v - 1}) \cdot (\frac{2v + 1}{12v}) \\
&\quad + (1 - \frac{36v^2 - 84v + 48}{v^2 - 3v}) \cdot (\frac{v}{24v - 24}) \cdot (\frac{v^2 + 2v}{12}) + (1 - \frac{36v^2 - 84v + 48}{v^2 - 3v}) \cdot (1 - \frac{v}{24v - 24}) \cdot (\frac{v + 1}{2v})
\end{aligned}$$

If $1.267 < v \leq 1.596$, by (7), (10), (16), (21), (22), (30), (31), (39) and (40) the players' expected total effort is

$$\begin{aligned}
TE^{RR} &= TE^{F3} + p_{12}^{F3} \cdot TE^{E2} + p_{21}^{F3} \cdot TE^{D2} + p_{12}^{F3} \cdot p_{31}^{E2} \cdot TE^{C} + p_{21}^{F3} \cdot p_{13}^{D2} \cdot TE^{B1} + p_{21}^{F3} \cdot p_{31}^{D2} \cdot TE^{A} \\
&= (\frac{v^3 - 3v^2 - 6v + 6}{24v^4 - 72v^3}) + (\frac{-3v + 3}{v^3 - 3v^2}) \cdot (\frac{144v^3 + 84v^2 - 20v + 1}{288v^3 - 96v^2}) \\
&\quad + (1 - \frac{-3v + 3}{v^3 - 3v^2}) \cdot (\frac{13v^2 - 12v}{288v - 288}) + (\frac{-3v + 3}{v^3 - 3v^2}) \cdot (1 - \frac{12v - 1}{24v^2 - 8v}) \cdot (\frac{2v + 1}{12v}) \\
&\quad + (1 - \frac{-3v + 3}{v^3 - 3v^2}) \cdot (\frac{v}{24v - 24}) \cdot (\frac{v^2 + 2v}{12}) + (1 - \frac{-3v + 3}{v^3 - 3v^2}) \cdot (1 - \frac{v}{24v - 24}) \cdot (\frac{v + 1}{2v})
\end{aligned}$$

If $1.596 < v \leq 2$, by (7), (10), (16), (21), (22), (30), (31), (42) and (43) the players' expected total effort is

$$\begin{aligned}
TE^{RR} &= TE^{F4} + p_{12}^{F4} \cdot TE^{E2} + p_{21}^{F4} \cdot TE^{D2} + p_{12}^{F4} \cdot p_{31}^{E2} \cdot TE^{C} + p_{21}^{F4} \cdot p_{13}^{D2} \cdot TE^{B1} + p_{21}^{F4} \cdot p_{31}^{D2} \cdot TE^{A} \\
&= (\frac{v^5 - 6v^4 + 3v^3 + 24v^2 - 18v}{864v^2 - 1728v + 864}) + (1 - \frac{v^3 - 3v^2}{-12v + 12}) \cdot (\frac{144v^3 + 84v^2 - 20v + 1}{288v^3 - 96v^2}) \\
&\quad + (\frac{v^3 - 3v^2}{-12v + 12}) \cdot (\frac{13v^2 - 12v}{288v - 288}) + (1 - \frac{v^3 - 3v^2}{-12v + 12}) \cdot (1 - \frac{12v - 1}{24v^2 - 8v}) \cdot (\frac{2v + 1}{12v}) \\
&\quad + (\frac{v^3 - 3v^2}{-12v + 12}) \cdot (\frac{v}{24v - 24}) \cdot (\frac{v^2 + 2v}{12}) + (\frac{v^3 - 3v^2}{-12v + 12}) \cdot (1 - \frac{v}{24v - 24}) \cdot (\frac{v + 1}{2v})
\end{aligned}$$

And if $v > 2$, by (7), (13), (16), (24), (25), (30), (31), (45) and (46) the players' expected total effort is

$$\begin{aligned}
TE^{RR} &= TE^{F5} + p_{12}^{F5} \cdot TE^{E2} + p_{21}^{F5} \cdot TE^{D3} + p_{12}^{F5} \cdot p_{31}^{E2} \cdot TE^{C} + p_{21}^{F5} \cdot p_{13}^{D3} \cdot TE^{B2} + p_{21}^{F5} \cdot p_{31}^{D3} \cdot TE^{A} \\
&= (\frac{4v^2 - 7v + 3}{48v^3 - 48v^2 + 12v}) + (1 - \frac{v - 1}{2v - 1}) \cdot (\frac{144v^3 + 84v^2 - 20v + 1}{288v^3 - 96v^2}) \\
&\quad + (\frac{v - 1}{2v - 1}) \cdot (\frac{2v^3 - 2v^2 - v + 1}{12v^3 - 6v^2}) + (1 - \frac{v - 1}{2v - 1}) \cdot (1 - \frac{12v - 1}{24v^2 - 8v}) \cdot (\frac{2v + 1}{12v}) \\
&\quad + (\frac{v - 1}{2v - 1}) \cdot (\frac{v - 1}{4v^2 - 2v}) \cdot (\frac{v + 2}{3v}) + (\frac{v - 1}{2v - 1}) \cdot (1 - \frac{v - 1}{4v^2 - 2v}) \cdot (\frac{v + 1}{2v})
\end{aligned}$$

11.3.2 Case B: The weak players are matched in the second stage (stage 1: 1 vs. 3, stage 2: 1 vs. 2, stage 3: 2 vs. 3)

In this case if $1 < v \leq 2$, by (49), (52), (55), (59), (60), (62), (63), (68) and (69) the players' expected total effort is

$$\begin{aligned}
TE^{RR} &= TE^{F1} + p_{13}^{F1} \cdot TE^{E1} + p_{31}^{F1} \cdot TE^{D} + p_{13}^{F1} \cdot p_{21}^{E1} \cdot TE^{C1} + p_{31}^{F1} \cdot p_{12}^{D} \cdot TE^{B} + p_{31}^{F1} \cdot p_{21}^{D} \cdot TE^{A} \\
&= (\frac{64v^4 - 16v^3 - 20v^2 + 4v + 1}{288v^3 - 96v^2}) + (\frac{4v^2 - 1}{24v^2 - 8v}) \cdot (\frac{-5v^2 + 39v - 72}{6v - 72}) \\
&\quad + (\frac{4v^2 - 1}{24v^2 - 8v}) \cdot (\frac{2v - 6}{v - 12}) \cdot (\frac{v^2 + 2v}{12}) + (1 - \frac{4v^2 - 1}{24v^2 - 8v}) \cdot (\frac{2v + 1}{12v})
\end{aligned}$$

And if $v > 2$, by (49), (52), (58), (59), (60), (65), (66), (71) and (72) the players' expected total effort is

$$\begin{aligned}
TE^{RR} &= TE^{F2} + p_{13}^{F2} \cdot TE^{E2} + p_{31}^{F2} \cdot TE^{D} + p_{13}^{F2} \cdot p_{21}^{E2} \cdot TE^{C2} + p_{31}^{F2} \cdot p_{12}^{D} \cdot TE^{B} + p_{31}^{F2} \cdot p_{21}^{D} \cdot TE^{A} \\
&= (\frac{176v^4 + 170v^3 - 8v^2 - 10v + 1}{528v^4 + 192v^3 - 96v^2}) + (\frac{16v^2 + 6v - 1}{44v^3 + 16v^2 - 8v}) \cdot (\frac{3v + 1}{12v + 6}) \\
&\quad + (\frac{16v^2 + 6v - 1}{44v^3 + 16v^2 - 8v}) \cdot (\frac{v}{4v + 2}) \cdot (\frac{v + 2}{3v}) + (1 - \frac{16v^2 + 6v - 1}{44v^3 + 16v^2 - 8v}) \cdot (\frac{2v + 1}{12v})
\end{aligned}$$

11.3.3 Case C: The weak players are matched in the third stage (stage 1: 1 vs. 3, stage 2: 2 vs. 3, stage 3: 1 vs. 2)

In this case if $1 < v \leq 1.45$, by (75), (78), (81), (82), (83), (85), (86), (88) and (89) the players' expected total effort is

$$\begin{aligned} TE^{RR} &= TE^{F1} + p_{13}^{F1} \cdot TE^{D} + p_{31}^{F1} \cdot TE^{E} + p_{13}^{F1} \cdot p_{23}^{D} \cdot TE^{A} + p_{13}^{F1} \cdot p_{32}^{D} \cdot TE^{B} + p_{31}^{F1} \cdot p_{23}^{E} \cdot TE^{C} \\ &= (\frac{121v^2 - 88v}{192}) + (\frac{11v-8}{16}) \cdot (\frac{11v+8}{33v}) + (1 - \frac{11v-8}{16}) \cdot \frac{1}{4} + (\frac{11v-8}{16}) \cdot (\frac{4}{11v}) \cdot \frac{1}{4} \end{aligned}$$

And if $v > 1.45$, by (75), (78), (81), (82), (83), (85), (86), (91) and (92) the players' expected total effort is

$$\begin{aligned} TE^{RR} &= TE^{F2} + p_{13}^{F2} \cdot TE^{D} + p_{31}^{F2} \cdot TE^{E} + p_{13}^{F2} \cdot p_{23}^{D} \cdot TE^{A} + p_{13}^{F2} \cdot p_{32}^{D} \cdot TE^{B} + p_{31}^{F2} \cdot p_{23}^{E} \cdot TE^{C} \\ &= (\frac{11v}{33v-24}) + (1 - \frac{4}{11v-8}) \cdot (\frac{11v+8}{33v}) + (\frac{4}{11v-8}) \cdot \frac{1}{4} + (1 - \frac{4}{11v-8}) \cdot (\frac{4}{11v}) \cdot \frac{1}{4} \end{aligned}$$