THE AXIOM OF EQUIVALENCE TO INDIVIDUAL POWER AND THE BANZHAF INDEX

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The Axiom of Equivalence to Individual Power and the Banzhaf Index*

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Abstract

I introduce a new axiom for power indices on the domain of finite simple games that requires the total power of any given pair i, j of players in any given game v to be equivalent to some individual power, i.e., equal to the power of *some* single player k in *some* game w. I show that the Banzhaf power index is uniquely characterized by this new "equivalence to individual power" axiom in conjunction with the standard semivalue axioms: transfer (which is the version of additivity adapted for simple games), symmetry or equal treatment, positivity (which is strengthened to avoid zeroing-out of the index on some games), and dummy.

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1 Introduction

The Shapley-Shubik power index (henceforth SSI) and the Banzhaf power index (henceforth BI) were devised as measures of the individual power in voting situations, and are based on simple computational formulas. The BI, introduced in

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Banzhaf (1965), is easier to describe – the voting power of a participant is defined as the probability that he is a "swinger", i.e., that his "yes" vote changes the voting outcome, when all other individuals cast their votes independently and with equal probability for "yes" and "no". If, as suggested by Shapley and Shubik (1954), the voting situation is modelled as a simple cooperative game with a finite player set N, the BI of player $i \in N$ is given by his probability to turn a random coalition of players from losing to winning by joining it, assuming that the coalition is chosen with respect to the uniform distribution over the subsets of $N \backslash \{i\}$.

Naturally, other probability distributions over the subsets of $N \backslash \{i\}$ can be considered, one of which leads to the famous SSI, introduced in Shapley and Shubik (1954). The SSI is induced by the uniform distribution over all strict orderings of N, and player i's voting power is defined as the probability of him being pivotal in a random order (that is, the probability that by joining the coalition of his predecessors in a random order, i switches it from losing to winning). A major distinctive feature of SSI is its *efficiency* – namely, the total power of all players is 1 in any simple game.[1] In contrast, BI is not efficient in general. The total power in a game is equal to the expected number of "swingers" in that game. The latter property has been elevated to the rank of an axiom by Dubey and Shapley (1979) as a substitute for efficiency, in an attempt to provide BI with an axiomatic foundation that would mirror that of SSI (established in Dubey (1975)).

The Dubey and Shapley axiom may be deemed unsatisfactory,[2] however, not least because it explicitly relies on counting "swings" (the notion on which BI is based). Fortunately, BI has other distinguishing features that can replace this axiom, of which we shall mention just two. The *composition* property, that was formally defined and proved by Owen (1975, 1978), pertains to a two-tier voting process, and requires the power of player i in a compound voting game to be equal[3] to the product of i's power in the first-tier game in which he participates and the power of i's delegate

[1] In this paper we adopt the Dubey and Shapley (1979) notion of a simple game, that requires the grand coalition N to be winning, and the game to be monotonic (i.e., a winning coalition must remain winning if joined by one or more players).

[2] See e.g. Section 5 in Dubey et al. (2005).

[3] To be precise, a second-tier game needs to be decisive (namely, a winning coalition must have a losing complement in N, and vice versa) for the composition property to hold.

in the second-tier game.[4] Another distinctive property of BI is *2-efficiency*. It was established in Lehrer (1988) and requires the sum of the power of any two players, i and j, in any game v to be equal to the power of player i in the game $v_{i,j}$ obtained from v by "merging" j into i (i.e., any coalition that contains i in the game $v_{i,j}$ has the same worth as the coalition that contains both i and j in the game v).

The 2-efficiency property is quite powerful. Lehrer (1988) showed that any 2-efficient power index that coincides with BI on the set of all 2-player simple games is, in fact, identical to the BI on all games. But 2-efficiency is also powerful enough to be a basis for an axiomatization of BI that does not contain an explicit or implicit comparison to BI on certain games. Lehrer (1988) considered a weaker version of 2-efficiency, which he termed the *superadditivity* axiom, whereby the total power of any i,j in any v *does not exceed* the power of the merged player i in $v_{i,j}$. Lehrer proved that BI is uniquely characterized by the superadditivity axiom, along with other requirements that are routinely imposed[5] on power indices (these are the *transfer*, *equal treatment* or *symmetry*, and *dummy* axioms).[6] Recently, Casajus (2012) showed that the symmetry axiom is not needed in Lehrer's characterization of BI (but the three remaining axioms are logically independent). That is, superadditivity, transfer and dummy axioms uniquely characterize BI on the set of finite simple games.

In this work we introduce a new axiom, *equivalence to individual power (EIP)*, that is related to 2-efficiency but has an independent conceptual appeal. The EIP axiom is based on the idea that when trying to conceptualize the collective power of a pair of players, one need not leave the realm where only the individual power is defined, as the collective power has an ordinal equivalent in that realm. Formally, EIP postulates that, given any two players $i, j \in N$ and a simple game v on N, the total power of the pair i, j in v is equivalent to some individual power, i.e., equal to the power of *some* (single) player k in *some* simple game w. Only one mild assumption

[4] A composition property-based axiomatization of the BI on the domain of simple games appeared in Dubey et al. (2005).

[5] See Dubey (1975), Dubey and Shapley (1979), Einy (1987).

[6] The superadditivity and 2-efficiency also figure prominently in axiomatizations of the Banzhaf *value* (BV), the extension of BI to the set of all games on N. Lehrer's (1988) Theorem B establishes a characterization of BV that is identical to that of BI, with the linearity axiom instead of transfer. See also the works of Nowak (1997) and Casajus (2011, 2012), where the linearity axiom is replaced by versions of Young's (1985) monotonicity.

links w to the original game v: w should have the same – or smaller – carrier compared to v. There need not be any other relation between w and v. Thus, according to EIP, the "language" of individual power must be sufficiently "expressive" to also be able to capture the total power of pairs; in mathematical terms, the union of the image sets of all players' individual power indices must be sufficiently rich so as to contain the image sets of all 2-sums of individual power indices.

The usual formulations of the 2-efficiency property (such as those in Lehrer (1988), Casajus (2012)) treat the original simple game v and the merged $v_{i,j}$ as having different player sets (with j missing from the player set of the latter). To allow comparison with EIP, we note that v and $v_{i,j}$ can be assumed to have the same set of players N, but different carriers: if v has a carrier $T \subset N$, then $T\backslash\{j\}$ acts as a carrier for $v_{i,j}$ (with j being a null player). With this convention, the axiomatization of BI in Lehrer (1988) holds with a fixed player set N.[7] The EIP axiom can thus be viewed as a weakening of the 2-efficiency property. Indeed, given a 2-efficient power index, a simple game v on N and $i, j \in N$, take w to be the merged game $v_{i,j}$ and k to be the merged player i; then any carrier of v is also a carrier of $v_{i,j} = w$, and, by 2-efficiency, $i = k$ has the same power as what i and j hold together in v. Note that the relaxation of 2-efficiency that is embodied in the EIP axiom is significant, because the latter is very permissive as to what individual power may be considered in seeking the match for the total power of i and j. The matching individual power is not required to be obtainable by merging i and j in v; in fact, it may come from a game not related to v in any way (other than having the same carrier).

Our main result (Theorem 1) is that BI is the only power index that satisfies the EIP axiom together with the standard set of four "semivalue axioms" (the term comes from Einy (1987), who calls power indices satisfying the four axioms *semivalues on simple games*). The set contains the transfer, symmetry, and dummy axioms that have already been mentioned, and the *positivity* axiom that, in addition to the standard requirement of non-negativity of the power index for all players, stipulates that positive power must be attributed to at least one player in a game.[8] We then show, using arguments similar to those of Malawski (2002), that a weaker equal

[7]The axiomatization in Theorem 5 of Casajus (2012) that removes the symmetry axiom from Lehrer's list also holds for games with a fixed player set N, assuming $|N| \geq 3$.

[8]Without the latter requirement, our result would not hold as is explained in Remark 2(4).

treatment property can replace the symmetry in our set of axioms (see Corollary 1).

Our five axioms are independent when there are at least three players (see Remark 2). Thus, the EIP axiom is "permissive" not only in appearance – it is strictly weaker than 2-efficiency because the latter uniquely characterizes BI combined with just three semivalue axioms (positivity excepted) by Lehrer (1988).[9] However, we also point out that the weakening of EIP that only bounds the total power of a pair from above by some individual power cannot replace EIP in our axiomatization (because then the SSI would satisfy all the axioms, see Remark 2(1)), unlike superadditivity in the axiomatization of Lehrer (1988) that was substitutable for the stronger 2-efficiency. Furthermore, the EIP axiom cannot be a basis for an axiomatic characterization of the Banzhaf value on the set of *all* games on N, as then the all-games version of EIP would be satisfied by every semivalue (see Remark 4).

2 Simple Games and the Banzhaf Index

Let $N = \{1, 2, ..., n\}$ be the *player set*, that will be fixed throughout. Denote the collection of all *coalitions* (subsets of N) by 2^N, and the empty coalition by $\varnothing$. Then a game on N (or simply a *game*) is given by a map $v : 2^N \to R$ with $v(\varnothing) = 0$. The space of all games is denoted by $\mathcal{G}(N)$. A coalition $T \in 2^N$ is called a *carrier* of v if $v(S) = v(S \cap T)$ for any $S \in 2^N$.

The domain $\mathcal{SG}(N) \subset \mathcal{G}(N)$ of *simple games* on N consists of all $v \in \mathcal{G}$ such that

(i) $v(S) \in \{0, 1\}$ for all $S \in 2^N$;

(ii) $v(N) = 1$;

(iii) v is *monotonic*, i.e., if $S \subset T$ then $v(S) \leq v(T)$.

A coalition S is said to be *winning* in $v \in \mathcal{SG}(N)$ if $v(S) = 1$, and *losing* otherwise. The set of simple games (respectively, all games) with carrier $T \subset N$ will be denoted by $\mathcal{SG}(T)$ (respectively $\mathcal{G}(T)$). Given a non-empty set $T \subset N$, denote by $u_T \in \mathcal{SG}(T)$ the unanimity game with carrier T, i.e., the game for which T is the (only) minimal winning coalition.

[9] In fact, as the above-mentioned result of Casajus (2012) holds when the player set N is fixed and $|N| \geq 3$, the transfer and dummy axioms alone can uniquely characterize the BI in conjunction with 2-efficiency.

A *power index* is a mapping $\varphi : \mathcal{SG}(N) \to \mathbb{R}^n$. For each $i \in N$ and $v \in \mathcal{SG}(N)$, the i^{th} coordinate of $\varphi(v) \in \mathbb{R}^n$, $\varphi(v)(i)$, is interpreted as the voting power of player i in the game v. The Banzhaf index (henceforth BI) and the Shapley-Shubik index (henceforth SSI) are among the best known power indices. In this article we focus on the former. The BI is given for each $v \in \mathcal{SG}(N)$ and $i \in N$ by

$$\beta(v)(i) = \sum_{S \subset N \setminus \{i\}} \frac{1}{2^{n-1}} \left[v(S \cup \{i\}) - v(S) \right]. \tag{1}$$

Thus, for each $i \in N$, $\beta(v)(i)$ is the expected marginal contribution of player i to a random coalition, chosen w.r.t. the uniform distribution on subsets of $N \setminus \{i\}$. Equivalently, $\beta(v)(i)$ is the probability that the random coalition is losing in v but becomes winning if joined by i. (The SSI is given by a modification of (1), where in each summand the coefficient $\frac{1}{2^{n-1}}$ is replaced by $\frac{|S|!(n-|S|-1)!}{n!}$.)

3 The Axioms

We shall show that β is the unique power index on $\mathcal{SG}(N)$ which satisfies the five axioms below. The first axiom is new. It postulates that the total power of a pair of players, as measured by the index, is equivalent to some individual power. To be precise, the sum of power indices of any two given players in a game is required to be equal to the power index of *some* player in *some* game with the same carrier (that need not be related in any other way to the original game).

Axiom I: Equivalence to individual power (EIP). If $T \subset N$ and $v \in \mathcal{SG}(T)$, then for every two distinct players $i, j \in T$ there exist $k \in T$ and $w \in \mathcal{SG}(T)$ such that

$$\varphi(v)(i) + \varphi(v)(j) = \varphi(w)(k). \tag{2}$$

EIP is related to the superadditivity (**SA**) and 2-efficiency (**2-EF**) axioms introduced by Lehrer (1988) for his characterization of the BI. Given $T \subset N$, $i, j \in T$ and $v \in \mathcal{SG}(T)$, the two axioms compare the sum $\varphi(v)(i) + \varphi(v)(j)$ with $\varphi(v_{i,j})(i)$, where $v_{i,j} \in \mathcal{SG}(T \setminus \{j\})$ is given by $v_{i,j}(S) = v(S \setminus \{j\})$ and $v_{i,j}(S \cup \{i\}) = v(S \cup \{i, j\})$ for any $S \subset N \setminus \{i\}$ (i.e., in $v_{i,j}$ the two distinct players i, j are "merged" into a single player, i). SA requires that

$$\varphi(v)(i) + \varphi(v)(j) \leq \varphi(v_{i,j})(i); \tag{3}$$

2-EF requires (3) to hold with equality. (In our rendering of **SA** and **2-EF** we follow the interpretation of Casajus (2012), in that the new entity created by the merger of i and j retains the name i; however, in order to keep the player set N unchanged, in accordance with our basic assumption, j is assumed not to leave the game but to remain as a null player[10].)

It is clear that **EIP** is weaker than **2-EF,** as an equality in (3) implies (2) by taking $w = v_{i,j}$ and $k = i$. Conceptually, **EIP** places almost no restriction on the source of an individual power needed to match the total power of i and j. Indeed, the individual power is not required to be obtainable by merging i and j in v, and may come from a game not related to v in any way, other than having the same carrier T. However, **EIP** and **SA** are incomparable. On the one hand, **EIP** requires the total power of a pair i, j to be precisely matched by the individual power of some player, while **SA** only requires the total power to be bounded from above by the individual power. On the other hand, **EIP** is significantly more permissive as to the source of the individual power.

Remark 1 (2-EF of the BI). By Proposition 1 in Lehrer (1988), β satisfies **2-EF**, and hence **SA**. (This implies that β also satisfies **EIP**.)

The four axioms that follow are standard, and their variants are present in the original axiomatizations of SSI and BI (see Dubey (1975) and Dubey and Shapley (1979)). Power indices that satisfy them were termed *semivalues* on $\mathcal{SG}(N)$ in Einy (1987). To state our second axiom, we introduce the following notation. For $v, w \in \mathcal{SG}(N)$ define $v \vee w, v \wedge w \in \mathcal{SG}(N)$ by:

$$(v \vee w)(S) = \max\{v(S), w(S)\},$$

$$(v \wedge w)(S) = \min\{v(S), w(S)\}$$

for all $S \in 2^N$. (It is evident that $\mathcal{SG}(N)$ is closed under operations $\vee, \wedge$.) Thus a coalition is winning in $v \vee w$ if, and only if, it is winning in at least one of v or w, and it is winning in $v \wedge w$ if, and only if, it is winning in both v and w.

[10] This term is formally defined in a discussion following the statement of Axiom V (**D**) below.

Axiom II: Transfer (T). $\varphi(v \vee w) + \varphi(v \wedge w) = \varphi(v) + \varphi(w)$ for all $v, w \in \mathcal{SG}(N)$.

As remarked in Dubey et al. (2005), **T** can be restated in the following equivalent form. Consider two pairs of games v, v' and w, w' in $\mathcal{SG}(N)$, and suppose that the transitions from v' to v and w' to w entail adding the *same* set of winning coalitions (i.e., $v \geq v'$, $w \geq w'$, and $v - v' = w - w'$). An equivalent axiom would require that

$$\varphi(v) - \varphi(v') = \varphi(w) - \varphi(w'),$$

i.e., that the change in power depends only on the change in the voting game.

Next, denote by $\Pi(N)$ the set of all *permutations* of N (i.e., bijections $\pi : N \to N$). For $\pi \in \Pi(N)$ and a game $v \in \mathcal{SG}(N)$, define $\pi v \in \mathcal{SG}(N)$ by

$$(\pi v)(S) = v(\pi(S))$$

for all $S \in 2^N$. The game πv is the same as v except that players are relabeled according to π.

Axiom III: Symmetry (S). $\varphi(\pi v)(i) = \varphi(v)(\pi(i))$ for every $v \in \mathcal{SG}(N)$, every $i \in N$, and every $\pi \in \Pi(N)$.

According to **S**, if players are relabeled in a game, their power indices will be relabeled accordingly. Thus, irrelevant characteristics of the players, outside of their role in the game v, have no influence on the power index.

Axiom IV: Positivity (P). $\varphi(v) \in \mathbb{R}^n_+$ and the vector $\varphi(v)$ is non-zero for each $v \in \mathcal{SG}(N)$.

The positivity requirement is natural, as it is hard to imagine how a negative power could be associated with a player who can never make matters worse by joining a group. Since every $v \in \mathcal{SG}(N)$ is monotonic by assumption, no player can indeed turn a winning coalition into a losing one by joining it. Similarly, as the grand coalition N is by assumption a winning coalition in any $v \in \mathcal{SG}(N)$, winning is possible, hence the expectation that *positive* power should be attributed to at least one player.

Axiom V: Dummy (D). If $v \in \mathcal{SG}(N)$ and i is a dummy player in v, i.e. $v(S \cup \{i\}) = v(S) + v(\{i\})$ for every $S \subset N \backslash \{i\}$, then $\varphi(v)(i) = v(\{i\})$.

This axiom can be viewed as a normalization requirement. If i does not belong to a carrier of $v \in \mathcal{SG}(N)$, he is a dummy with $v(\{i\}) = 0$, i.e. a *null player*. He changes nothing by joining a coalition, and hence in the distribution of power it is convenient to view his share as the lowest possible, 0. On the other hand, if i is a dummy player in v and $v(\{i\}) = 1$, then he is a *dictator*, namely, a coalition is winning if and only if it contains i. In terms of power distribution, the power attributed to him should be at the highest possible level, and it is convenient to label this level as 1 $(= v(\{i\}))$.

4 The Results

Our main result is the following theorem.

Theorem 1. There exists one, and only one, power index satisfying **EIP, T, S, P,** and **D**, and it is the BI.

Proof. As mentioned in Remark 1, β satisfies **EIP**, and it is well known that β also satisfies the semivalue axioms **T, S, P,** and **D**. It remains to show that our five axioms uniquely determine β. To this end, fix any power index φ that satisfies **EIP, T, S, P,** and **D**. On account of the last four axioms, φ is a semivalue on $\mathcal{SG}(N)$, and so Theorem 2.4 of Einy (1987) applies[11]. Thus, there is a unique extension of φ to $\overline{\varphi} : \mathcal{G}(N) \to \mathbb{R}^n$ that is a *semivalue* on $\mathcal{G}(N)$ (the latter means that (i) $\overline{\varphi}$ is a linear map; (ii) $\overline{\varphi}$ satisfies the symmetry axiom for any $v \in \mathcal{G}(N)$; (iii) $\overline{\varphi}(v)(i) = v(\{i\})$ for any additive[12] game $v \in \mathcal{G}(N)$ and $i \in N$; and (iv) $\varphi(v) \in \mathbb{R}^n_+$ for any monotonic game $v \in \mathcal{G}(N)$). The restriction of $\overline{\varphi}$ to the space $\mathcal{G}(T)$ for any $T \subset N$ obviously remains a semivalue on $\mathcal{G}(T)$. The Lemma in Dubey et al. (1981) that characterizes all semivalues on finite games can now be used to explicitly describe the restriction of $\overline{\varphi}$ to $\mathcal{G}(T)$ for any fixed $T \subset N$, of size $1 \leq |T| = t \leq n$: there exists a unique

[11] Although Theorem 2.4 of Einy (1987) is stated for semivalues on the set of finite simple games with an *infinite* universe of players U, its proof does not use the assumption that U is infinite.

[12] A game $v \in \mathcal{G}(N)$ is additive if $v(S) = \sum_{i \in S} v(\{i\})$ for every $S \subset N$.

collection $(p^t_s)_{s=0}^{t-1}$ of non-negative numbers satisfying

$$\sum_{s=0}^{t-1} \binom{t-1}{s} p^t_s = 1, \tag{4}$$

such that for every $v \in \mathcal{G}(T)$

$$\overline{\varphi}\,(v)\,(i) = \sum_{S \subset T \setminus \{i\}} p^t_{|S|}\,(v(S \cup \{i\}) - v(S)). \tag{5}$$

(By the symmetry of $\overline{\varphi}$ and the fact that $(p^t_s)_{s=0}^{t-1}$ is determined uniquely, the collection $(p^t_s)_{s=0}^{t-1}$ is independent of the choice of T with $|T| = t$.) In particular, for every $v \in \mathcal{SG}(N)$ with carrier $T \subset N$ of size t,

$$\varphi\,(v)\,(i) = \sum_{S \subset T \setminus \{i\}} p^t_{|S|}\,(v(S \cup \{i\}) - v(S)). \tag{6}$$

Now fix t, $1 \leq t \leq n-1$. Since $\mathcal{G}(\{1,2,...,t\}) \subset \mathcal{G}(\{1,2,...,t+1\})$, (5) can be applied (with $t+1$ instead of t) to any $v \in \mathcal{G}(\{1,2,...,t\})$ when the latter is viewed as a game with carrier $T = \{1,2,...t+1\}$. This yields

$$\overline{\varphi}\,(v)\,(i) = \sum_{S \subset \{1,...,t\} \setminus \{i\}} \left[p^{t+1}_{|S|} + p^{t+1}_{|S|+1}\right](v(S \cup \{i\}) - v(S)).$$

But as the collection $(p^t_s)_{s=0}^{t-1}$ is determined uniquely, it follows that

$$p^{t+1}_s + p^{t+1}_{s+1} = p^t_s \text{ for every } s = 0,...,t-1. \tag{7}$$

We will next show by induction on t that

$$p^t_s = \frac{1}{2^{t-1}} \text{ for every } s = 0,1,...,t-1. \tag{8}$$

For $t = 1$ the claim follows from (4), as then $p^1_0 = 1$. Now assume that (8) has been established for $t = m$, $1 \leq m < n$. We will show that it also holds for $t = m+1$. Combining the induction hypothesis (8) with (7) for $t = m$ yields

$$p^{m+1}_s + p^{m+1}_{s+1} = \frac{1}{2^{m-1}} \text{ for every } s = 0,...,m-1.$$

This, in turn, implies the following equalities:

$$\begin{aligned} p^{m+1}_0 &= p^{m+1}_2 = ... =: p, \\ p^{m+1}_1 &= p^{m+1}_3 = ... =: q, \text{ where } p+q = \frac{1}{2^{m-1}}. \end{aligned} \tag{9}$$

Consider two games in $\mathcal{SG}(\{1,2,...,m+1\})$,

$$v_1 = u_{\{1\}} \vee u_{\{2,...,m+1\}}$$

and

$$v_2 = u_{\{1,2\}} \vee ... \vee u_{\{1,m+1\}},$$

where (recall) u_T denotes the unanimity game with carrier T. Using (6) and (4) for $t = m+1$, we obtain

$$\varphi(v_1)(1) = \sum_{s=0}^{m-1} \binom{m}{s} p_s^{m+1} = 1 - p_m^{m+1},\ \varphi(v_1)(2) = p_{m-1}^{m+1}, \tag{10}$$

and

$$\varphi(v_2)(1) = \sum_{s=1}^{m} \binom{m}{s} p_s^{m+1} = 1 - p_0^{m+1},\ \varphi(v_1)(2) = p_1^{m+1}. \tag{11}$$

By **EIP**, there exist games $v_3, v_4 \in \mathcal{SG}(\{1,2,...,m+1\})$ and players $k, l \in \{1,2,...,m\}$ such that

$$\varphi(v_1)(1) + \varphi(v_1)(2) = \varphi(v_3)(k) \tag{12}$$

and

$$\varphi(v_2)(1) + \varphi(v_2)(2) = \varphi(v_4)(l).$$

It is immediate from (6) and (4) that $\varphi(v_3)(k), \varphi(v_4)(l) \leq 1$. Hence,

$$\varphi(v_1)(1) + \varphi(v_1)(2) \leq 1 \text{ and } \varphi(v_2)(1) + \varphi(v_2)(2) \leq 1.$$

Using (10) and (11), we obtain

$$1 - p_m^{m+1} + p_{m-1}^{m+1} \leq 1 \text{ and } 1 - p_0^{m+1} + p_1^{m+1} \leq 1. \tag{13}$$

In what follows we consider two cases.

Case 1: The integer m is odd. Using the notation in (9), the inequalities in (13) become $1 - q + p \leq 1$ and $1 - p + q \leq 1$, and hence $p = q$. The equalities in (9) now establish (8) for $t = m+1$.

Case 2: The integer m is even. Using the notation in (9), both inequalities in (13) lead to a single inequality

$$1 - p + q \leq 1. \tag{14}$$

Assume first that $1-p+q<1$. Consider again the games $v_1, v_3 \in \mathcal{SG}(\{1,2,...,m+1\})$ used above. Using (10), the notation in (9), and (12), we obtain

$$\varphi\left(v_3\right)(k)=1-p+q<1. \tag{15}$$

This means that

$$v_3\left(\{k\}\right)=0 \text{ or } v_3(N\backslash\{k\})=1, \tag{16}$$

since otherwise k would be a dictator (who is also a dummy player) in the monotonic game v_3, and by Axiom **D** he would receive $\varphi\left(v_3\right)(k)=v_3\left(\{k\}\right)=1$ in contradiction to (15). But then, using (6) and (4) for $t=m+1$,

$$\varphi\left(v_3\right)(k) \leq \sum_{s=1}^{m}\binom{m}{s} p_s^{m+1}=1-p_0^{m+1}=1-p$$

if $v_3\left(\{k\}\right)=0$, and

$$\varphi\left(v_3\right)(k) \leq \sum_{s=0}^{m-1}\binom{m}{s} p_s^{m+1}=1-p_m^{m+1}=1-p$$

if $v_3(N\backslash\{k\})=1$. Hence

$$\varphi\left(v_3\right)(k) \leq 1-p \tag{17}$$

no matter which of the two equalities in (16) hold. Since $q \geq 0$ by its definition in (9) and non-negativity of the coefficients $\left(p_s^{m+1}\right)_{s=0}^{m}$, the combination of (15) and (17) yields $q=0$.

Consider the 2-majority game $v^{2,m+1} \in \mathcal{SG}(\{1,2,...,m+1\})$ in which the set of minimal winning coalitions consists of all subsets of $\{1,2,...,m+1\}$ of size 2. Then, by (6) and the notation in (9), for every $i=1,2,...,m+1$,

$$\varphi\left(v^{2,m+1}\right)(i)=m p_1^{m+1}=mq=0.$$

Therefore $\varphi\left(v^{2,m+1}\right)$ is the zero vector, which contradicts the second requirement in Axiom **P**. This shows that there can be no strict inequality in (14), and hence $1-p+q=1$, or $p=q$. The equalities in (9) now establish (8) for $t=m+1$.

The treatment of cases 1 and 2 completes the induction step, and establishes (8) for every $t \leq n$, and in particular for $t=n$. The combination of (6) and (8) for $t=n$ and a comparison with (1) now shows that $\varphi=\beta$. ■

Remark 2 (Independence of the axioms when $n \geq 3$). Our axioms, **EIP, T, S, P,** and **D,** are independent when there are at least three players, as we show below.

1. The SSI satisfies all the axioms except **EIP**. Notice also that **EIP** cannot be weakened by requiring inequality "$\leq$" in (2) instead of equality.[13] Indeed, the SSI satisfies this weaker version of **EIP**, and so the weaker version, combined with the other axioms, would not uniquely characterize β.

2. Consider a power index φ on $\mathcal{SG}(N)$ that is equal to β for all games in $\mathcal{SG}(N)$, with the exception of $v = u_{\{1,2\}}$, for which $\varphi(v) = \frac{1}{2}\beta(v)$. The index φ satisfies all the axioms except **T**.

3. Let φ be given, for any $v \in \mathcal{SG}(N)$, by

$$\varphi(v)(i) = v(\{1,...,i\}) - v(\{1,...,i-1\})$$

if $i > 1$, and $\varphi(v)(1) = v(\{1\})$. (In particular, $\varphi(v)(i) \in \{0,1\}$ for every $i \in N$, and φ is efficient.) The index φ satisfies all the axioms except **S**.

4. Let φ be given, for any $v \in \mathcal{SG}(N)$ and $i \in N$, by

$$\varphi(v)(i) = \frac{1}{2}v(\{i\}) + \frac{1}{2}(v(N) - v(N \backslash \{i\})).$$

It is easy to see that φ satisfies **EIP, T, S,** and **D**. However, φ violates **P**. To see this, notice that due to the assumption that $n \geq 3$, $\varphi(v)$ is the zero vector for $v = v^{2,n}$, where $v^{2,n}$ is the 2-majority game supported on N (in which the minimal winning coalitions are precisely those of size 2). Notice further that $\varphi(v) \in \mathbb{R}^n_+$ for every $v \in \mathcal{SG}(N)$. This shows that **P** cannot be weakened by removing from it the requirement that $\varphi(v)$ is always non-zero, as this weaker version, combined with the other axioms, would not uniquely characterize the BI.

5. The power index $\varphi = \frac{1}{2}\beta$ satisfies all the axioms except **D**.

[13] Had **EIP** been stated with inequality "$\leq$" in (2), it would be implied (i.e. be weaker than) the **SA** axiom of Lehrer (1988).

Remark 3 (Redundancy of some axioms when $n \leq 2$). When $n = 1$, there is only one power index on (the unique game in) $\mathcal{SG}(\{1\})$, by **D**. When $n \geq 2$, Axiom **P** can be dropped from the list of axioms characterizing β. Indeed, take any power index φ on $\mathcal{SG}(\{1,2\}) = \{u_{\{1\}}, u_{\{2\}}, u_{\{1,2\}}, u_{\{1\}} \vee u_{\{2\}}\}$. By **D**, $\varphi\left(u_{\{1\}}\right) = \beta\left(u_{\{1\}}\right) = (1,0)$ and $\varphi\left(u_{\{2\}}\right) = \beta\left(u_{\{2\}}\right) = (0,1)$, and by **S** $\varphi\left(u_{\{1,2\}}\right) = (a,a)$ and $\varphi\left(u_{\{1\}} \vee u_{\{2\}}\right) = (b,b)$ for some a, b. By **T**,

$$(1,1) = \varphi\left(u_{\{1\}}\right) + \varphi\left(u_{\{2\}}\right) = \varphi\left(u_{\{1\}} \vee u_{\{2\}}\right) + \varphi\left(u_{\{1,2\}}\right) = (a+b, a+b),$$

hence $a + b = 1$. If $a \leq 0$ then $b \geq 1$, and φ fails **EIP** for the game $v = u_{\{1\}} \vee u_{\{2\}}$ because the total power of players 1 and 2, $2b \geq 2$, cannot be matched by any individual power. We conclude that $a > 0$, and, similarly, that $b > 0$. Thus φ does in fact satisfy **P**, and so by our theorem $\varphi = \beta$.

Remark 4 (Theorem 1 cannot be extended to axiomatize the Banzhaf value on the set of all games). Formula (1), when applied to every $v \in \mathcal{G}(N)$, defines the *Banzhaf value* $\overline{\beta} : \mathcal{G}(\mathcal{N}) \to \mathbb{R}^n$ on the entire $\mathcal{G}(N)$. However, **EIP**, were it to be stated for all games in $\mathcal{G}(N)$ and not just for simple games, would lose all its strength. Indeed, for any semivalue $\overline{\varphi}$ defined by (5) (for $t = n$ and a collection $(p_s^n)_{s=0}^{n-1}$ that is subject to (4)), the range of the individual value mapping $\overline{\varphi}(w)(k)$ – with variable $w \in \mathcal{G}(N)$ and fixed $k \in N$ – is the entire $\mathbb{R}$.[14] Thus, given any $v \in \mathcal{G}(N)$ and $i, j \in N$, for any semivalue $\overline{\varphi}$ there exists a game $w^{\overline{\varphi}} \in \mathcal{G}(N)$ such that $\overline{\varphi}(v)(i) + \overline{\varphi}(v)(j) = \overline{\varphi}(w^{\overline{\varphi}})(k)$. This shows that adding **EIP** to the set of semivalue axioms (which are extensions of **T, S, P,** and **D** to solutions on $\mathcal{G}(N)$) will not in any way narrow down the set of semivalues.

We shall finally note that in our axiomatization of BI Axiom **S** can be replaced by its following well-known weaker version:

Axiom VI: Equal Treatment (ET). If $i,j \in N$ are substitute players in the game $v \in \mathcal{SG}(N)$, i.e., for every $S \subset N \backslash \{i,j\}$ $v(S \cup \{i\}) = v(S \cup \{j\})$, then $\varphi(v)(i) = \varphi(v)(j)$.

[14] This is so even if the worth of the grand coalition, $w(N)$, was assumed to be fixed at some level, say $w(N) = v(N)$.

While **S** postulates that irrelevant characteristics of the players, outside of their role in the game v, have no influence on a power index, the weaker **ET** merely forbids discrimination between substitute players (with the same role in the game). The reason that **ET** can replace the stronger **S** is our next Proposition 1. The proposition is close, in statement and proof, to the conflation of Lemma 5 and Theorem 4(a) of Malawski (2002), who showed that **ET** is equivalent to **S** for any a linear value on $\mathcal{G}(N)$ that satisfies **D**. Malawski's result cannot, however, be directly imported into our setting of power indices on $\mathcal{SG}(N)$, as the linearity or additivity requirements are not applicable to power indices.

Proposition 1. Suppose that a power index φ satisfies **T, D,** and **ET.** Then φ also satisfies **S.**

Proof. We will make use of the following lemma.

Lemma 1. Given $S, T \subset N$ such that $|S| = |T| \geq 1$,

$$\varphi(u_S)(i) = \varphi(u_T)(j) \text{ for any } i \in S \text{ and } j \in T, \tag{18}$$

and

$$\varphi(u_S)(i) = \varphi(u_T)(j) = 0 \text{ for any } i \in N \backslash S \text{ and } j \in N \backslash T. \tag{19}$$

Proof of Lemma 1. Equality (19) follows from **D** as any player outside some carrier of a game is a dummy (null) player.

Denote $s := |S| = |T|$. If $S = T$ then (18) is implied by **ET**. Consider next the case where $S \neq T$ but the two sets have all but two players in common (i.e. $|S \cap T| = s - 1$), which means that $S = R \cup \{i'\}$ and $T = R \cup \{j'\}$ for some $i' \in N$, $j' \in N$, and $R \subset N \backslash \{i', j'\}$. As all $\varphi(u_S)(i)$ (respectively, all $\varphi(u_T)(j)$) are equal for $i \in S$ (respectively, $j \in T$) by **ET**, in order to establish the equality in (18) it suffices to show that $\varphi(u_S)(i') = \varphi(u_T)(j')$. By **T,**

$$\varphi(u_S) + \varphi(u_T) = \varphi(u_S \vee u_T) + \varphi(u_S \wedge u_T). \tag{20}$$

In both $u_S \vee u_T = u_{R \cup \{i'\}} \vee u_{R \cup \{j'\}}$ and $u_S \wedge u_T = u_{R \cup \{i', j'\}}$ the players i' and j' are substitutes, and hence by **ET** and (20)

$$\varphi(u_S)(i') + \varphi(u_T)(i') = \varphi(u_S)(j') + \varphi(u_T)(j').$$

Using (19), the above equality turns into $\varphi(u_S)(i') = \varphi(u_T)(j')$, which establishes (18) when $|S \cap T| = s - 1$. For general S and T of the same size s, a chain of s-sized coalitions between S and T can be found such that any two consecutive coalitions in the chain have all but two players in common (and to whom the above argument applies), and hence (18) holds for any S and T of size s. □

Now let $\overline{\varphi}$ be the power index given by

$$\overline{\varphi}(v)(i) \equiv \frac{1}{n!} \sum_{\pi \in \Pi(N)} \varphi(\pi v)\left(\pi^{-1}(i)\right)$$

for every $v \in \mathcal{SG}(N)$ and $i \in N$. It is easy to check that $\overline{\varphi}$ satisfies **S**, and it also satisfies **T** as φ does so. By its definition and the properties established in Lemma 1, $\overline{\varphi}$ coincides with φ on the collection $(u_T)_{\varnothing \neq T \subset N}$ of all unanimity games in $\mathcal{SG}(N)$. Any $v \in \mathcal{SG}(N)$ can be written as a maximum of a finite number of unanimity games:

$$v = u_{T_1} \vee u_{T_2} \vee ... \vee u_{T_k},$$

where $T_1, ..., T_k$ are the minimal winning coalitions in v. By Lemma 2.3 of Einy (1987), for any power index ψ that satisfies **T**

$$\psi(v) = \sum_{I \subset \{1,...,k\}, I \neq \varnothing} (-1)^{|I|+1} \psi\left(u_{\cup_{m \in I} T_m}\right), \tag{21}$$

and by applying (21) to φ, $\overline{\varphi}$ and using the fact that both coincide on all unanimity games, we obtain $\varphi(v) = \overline{\varphi}(v)$. As this holds for every $v \in \mathcal{SG}(N)$, $\varphi = \overline{\varphi}$. But $\overline{\varphi}$ satisfies **S** as has been noted earlier. ■

It is an immediate corollary of Theorem 1 and Proposition 1 that **ET** can replace **S** in our axiomatization of BI:

Corollary 1. BI is the only power index that satisfies **EIP, T, ET, P,** and **D**.

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