MEAN-EXTENDED GINI PORTFOLIOS: THE ULTIMATE FRONTIER

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Abstract

Using numerical optimization techniques we construct the mean-extended Gini (MEG) efficient frontier as a workable alternative to the mean-variance efficient frontier. The MEG model enables the introduction of specific risk aversion in portfolio selection and thus offers an alternative approach for calculating efficient portfolios and pricing risky assets. The resulting portfolios are stochastically dominant (SSD) for all risk-averse investors. Solving for MEG portfolios allows investors to construct efficient portfolios that are tailored to specific risk requisites. As a measure of risk, the model uses the extended Gini which is calculated by the covariance of asset returns with a weighing function of the cumulative distribution function (CDF) of these returns. Efficient MEG portfolios are obtained by minimizing the extended Gini of portfolio returns subject to a required mean return constraint. In a sample of asset returns, the CDF is estimated by ranking the returns. In this case analytical optimization techniques using continuous gradient approaches are unavailable, thus the need to develop numerical optimization techniques. In this paper we solve for MEG efficient portfolios expanding spreadsheet (*Excel*) techniques. In addition, using *Mathematica* software we develop a numerical optimization algorithm that finds the portfolio optimal frontier for arbitrarily large sets of shares. The result is a 3-dimension MEG efficient frontier in the mean, the extended Gini, and the risk aversion coefficient space.

Keywords: Mean-Gini portfolios, numerical optimization, stochastic dominance portfolios, 3D efficient frontier

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1. Introduction

The mean-Gini (MG) investment model offers an alternative to the standard mean-variance (MV) model by measuring risk using Gini's mean difference instead of the standard deviation. The MG and mean-extended-Gini (MEG) approaches were first introduced in finance by Shalit and Yitzhaki (1984). These models present an approach that prices risky assets and constructs efficient portfolios which are second-degree stochastic dominant (SSD) for all risk-averse investors. The MEG model developed earlier by Yitzhaki (1983) allowed for the specific introduction of risk aversion differentiation into the portfolio allocation process. Later, Shalit and Yitzhaki (2005) provided superior alternative optimal allocations to the MV efficient frontier, in particular when risky assets are not normally distributed. Although the methods were not initially adopted, they were later incorporated to many research projects and scientific articles.[1] Today, the MEG model is recognized for embodying a complete financial investment theory with econometric testing and for establishing a working environment for investments that are compatible with SSD and expected utility maximization. Our endeavor here is to synthesize MEG into a practical and complete form for use by both financial practitioners and academicians that extends its theoretical foundations to the working mechanics of portfolio allocation.

In this paper, we present numerical optimization algorithms in *Excel and Mathematica* in order to construct mean-extended Gini efficient portfolios for large sets of assets with and without short-sales positions. The purpose is to familiarize the investment practitioner with the MEG model as a substitute to the MV model in portfolio selection. Solving for MEG portfolios enables investors to construct efficient portfolios that are tailored to their specific risk requirements or to the needs of their clients. Indeed, when investors desire to hold riskier or less risky assets, MEG has the advantage of incorporating individual risk aversion. However, because the extended Gini is calculated by weighing the cumulative distribution function, analytical optimization techniques are unavailable. Hence, there is a need to develop numerical optimization techniques that would make MEG a superior tool for choosing optimal portfolios.

[1] See Bey and Howe (1984), Carroll et al(1992), Lien and Luo(1993), Okunev (1991) and Shalit(1995)

2. The Mean-Gini Investment Model

We present a portfolio investment model in which investors minimize their risk as measured by the Gini, subject to given expected returns. The Gini index is a statistic of dispersion used mainly in income distribution in order to compute income inequality. In financial economics the Gini index was first used by Fisher and Lorie (1970) to study the relative variability of single stocks and portfolios. In 1912, Corrado Gini developed a ratio to measure the relative variability of random variables. For a risky asset *x*, the numerator of the Gini index G_x is the expected absolute difference between all observations pairs, also called Gini's mean difference, $\Gamma_x = \frac{1}{2} \underset{i,j}{E} | x_i - x_j |$ where x_i and x_j are i.i.d. replicates of the risky asset *x* and the denominator of G_x is the mean of *x*, $E(x)$. In financial economics, it is customary to use as measure of risk the absolute value of the Gini index, called the Gini. This is an attractive measure as Γ_x evaluates variability by the absolute distance between all the data points instead of calculating their squared differences from a virtual central value like the mean.

The Gini Γ_x has many representations and formulations, most of which can be found in Yitzhaki (1998). The main feature of the Gini is that it estimates the pure risk of *x* since it can be obtained from the generalized (absolute) Lorenz curve of asset *x*. To see this, assume that asset *x's* returns are distributed by the cumulative probability distribution function (*CDF*), $F(x)$. Following Gastwirth (1971), the absolute Lorenz curve is defined over the probability φ as:

$$L_x(\varphi) = \int_0^{\varphi} F^{-1}(t)dt \qquad \text{for } 0 \le \varphi \le 1 \tag{1}$$

where $F^{-1}(t) = Inf\{x \mid F(x) \ge t\}$ is the inverse of the *CDF*. The absolute Lorenz curve of asset *x* is delineated in Figure 1. The curve starts at the origin (φ = 0) and ends up at the mean $E(x)$ for φ= 1. For identical means, the more convex the curve is, the riskier is the asset. The absolute Lorenz curve for the safest asset given a mean return is expressed by the straight line, called the line of safe asset (*LSA*), that runs from the origin to the mean of asset *x*. The pure risk of the asset is calculated by the area between the *LSA* that yields the same mean return and its absolute Lorenz curve since, for every probability φ, had one invest in the risky asset, one would get the cumulative expected return along the absolute Lorenz curve whereas investing in the riskless asset one would obtain a higher cumulative expected return on the *LSA*. Therefore, the farther the *LSA* is from the absolute Lorenz curve, the greater is the risk assumed by the asset.

In financial applications, the Gini is more conveniently expressed as $\Gamma_x = 2\operatorname{cov}[x, F(x)]$; i.e., the covariance between asset returns x and their cumulative probability distribution $F(x)$. The latter is estimated by i/T, which is the relative ranking of x_i sorted from the lowest return (i=1) to the highest (i=T).

As a measure to value uncertainty in risky assets, the Gini has been shown by Yitzhaki (1982) to have many advantages over the variance. Firstly, when returns depart from normality the Gini exhibits a better picture of the dispersion of the distribution since it compares the spread of observations among themselves. Secondly, together with the expected return, the Gini provides necessary and sufficient conditions for second degree stochastic dominance (*SSD*), implying that the mean and the Gini are a two parameter approach that is fully compatible with expected utility maximization. This feature is nonexistent with the variance and the mean unless asset returns are normally distributed or investor's preferences are quadratic. The mean-Gini conditions for *SSD* are stated as following:

Let x and y be two risky assets with means $E(x)$ and $E(y)$, and Gini's Γ_x and Γ_y respectively. Then, $E(x) \geq E(y)$ and $E(x) - \Gamma_x \geq E(y) - \Gamma_y$ are necessary and sometimes sufficient conditions for asset x to be preferred to asset y for all maximizing expected utility risk-averse investors.

Let us now present the mean-Gini (*MG*) portfolio model where investors use the mean and the Gini to construct *SSD* efficient portfolios by choosing the shares of risky assets that minimize the portfolio Gini subject to a given expected return. We consider a portfolio of N assets the weights of which are given as w_i and the portfolio returns p of which are given as $p = \sum_{i=1}^{N} w_i x_i$, where x_i are assets returns. The assets form a portfolio when the shares add up to on or : $\sum_{i=1}^{N} w_i = 1$. The investor has to choose the shares w_i that minimize the portfolio Gini $\Gamma_p = 2\operatorname{cov}[p, F_p(p)]$ subject to a given mean return constraint $E(p) = \sum_{i=1}^{N} w_i E(x_i)$ and subject or not to short sales restrictions $w_i \geq 0$. By introducing a riskless asset with safe return r_f, the mean return constraint becomes:

$$E(p) = r_f + \sum_{i=1}^{N} w_i \left(E(x_i) - r_f \right) \qquad (2)$$

In theory, the MG model is similar to the MV model and would yield identical results if asset returns were normally distributed. For any other probability distribution, the analyst is expected to obtain different allocation results. An additional advantage of using the Gini as measure of risk is that MG can be extended into a family of increasingly risk-averse models as developed by Yitzhaki (1983). Indeed, by adding one extra parameter, ν>1, the extended Gini measures asset risk when the lower portions of the returns distribution are multiplied by larger relative weights that express the concern investors have for losses when investing in risky assets. To specify increasing risk aversion, we derive the extended-Gini statistic by stressing the lower segments of the distribution of asset returns. As we recall from Figure 1, the simple Gini is obtained by calculating the area between the *LSA* and the Lorenz curve. Similarly, we obtain the extended Gini by adding the relatively *weighted* vertical differences between the *LSA* and the Lorenz curve. This area is calculated using the parameter ν to obtain the extended Gini of asset *x* as follows:

$$\Gamma_x(\nu) = \nu(\nu\text{-}1)\int_0^1 (1\text{-}\varphi)^{\nu\text{-}2}(\varphi E(x) - L_x(\varphi))d\varphi \qquad (3)$$

where $L_x(\varphi)$ is the Lorenz curve from Equation (1), $\varphi E(x)$ is the *LSA*, and $\nu(\nu\text{-}1)(1-\varphi)^{\nu-2}$ are the weights associated with each portion of the area between the *LSA* and the Lorenz curve. The parameter ν (>0) is the risk aversion coefficient chosen by analysts to represent the relative fear of losses by investors. Some special cases of interest for the extended Gini parameter include the following: For ν = 2 Equation (3) becomes the simple Gini. For ν → ∞ the extended Gini reflects the attitude of a max-min investor who expresses risk only in terms of the worst outcome. For ν → 1, Equation (3) cancels out, allowing risk-neutral investors without measures of dispersion to evaluate risk. For 0< ν <1 the extended Gini is negative and relates to risk-loving investors. For ease of presentation and because we are dealing with risk-averse investors, we consider here only the extended Gini with ν > 1, although many of the results can be applied without modification to risk-loving investors. In financial analysis, it is easier to express the extended Gini using the covariance formula rather than Equation(3):

$$\Gamma_x(\nu) = -\nu \operatorname{cov}\{x, [1 - F(x)]^{\nu-1}\} \quad (4)$$

To understand the essence of risk aversion using the extended Gini, the reader is referred to the swimmer/ shark metaphor from Shalit and Yitzhaki (2009, p. 761): "…. As an example, imagine a shark is roaming the coastal waters. A risk-neutral swimmer will calculate the swimming benefits by using the objective probability of being struck by a shark. If the swimmer uses ν=2, she will attach as the probability of being struck, twice her entrance into the water although she will jump only once. If the swimmer uses ν→∞, although she intends to enter the water only once, her behavior is as if she will be entering an infinite number of times. That is, if there is a tiny objective probability of having a shark roaming the waters, the behavior of the ν→∞ swimmer is as if the shark will strike with a probability of one…." Hence, we can see that the parameter ν and the extended Gini span an entire continuous spectrum of risk-aversion behavior

Moreover, with the extended Gini, *CAPM*s can be estimated, for each ν, as has been done in studies on equity and futures markets (see Gregory-Allen and Shalit, 1999, Lien and Luo, 1993, and Shalit, 1995, to cite just a few). The main econometric results of these papers show that when Ordinary Least-Squares are used as estimators, unwarranted sensitivity of *CAPM* betas is to be expected because of the fat tails of market returns distribution. *MEG* estimation has been shown to resolve some of these anomalies. *MEG* not only improves the quality of estimators by making them more robust to outliers, it also presents a framework to compare results with *MV* when market returns are not normal (see Carroll, Thistle, and Wei, 1992, Shalit and Yitzhaki, 2002).

From an investor's point of view, given a specific ν, efficient portfolio frontiers can be constructed with and without allowing for short sales. This permits investors to construct efficient portfolios that are tailored to their specific risk needs. Indeed, when investors desire to hold riskier or less risky assets, *MEG* has the advantage of incorporating individual risk- aversion in the choice process itself without relying on the portfolio separation theorem. The investor's problem is to choose the positions that minimize the extended Gini of a portfolio of assets subject to a given mean as follows:

Consider a portfolio p of N assets whose weights are w_i and whose returns are given by $r_p = \sum_i^n w_i r_i$, where r_i are the assets' returns. A portfolio of assets requires that $\sum_{i=1}^{N} w_i = 1$. Hence,

$$\begin{aligned} &\text{Minimize} \quad -\nu \sum_{i=1}^{N} w_i \operatorname{cov}\{x_i, [1 - F_p(p)]^{\nu-1}\} \\ &\text{subject to} \quad E(p) = \sum_{i=1}^{N} w_i E(x_i) \\ &1 = \sum_{i=1}^{N} w_i \end{aligned} \tag{5}$$

Changing the required mean allows the financial analyst to span the entire efficient frontier. The advantage of *MEG* is rooted in the different number of efficient frontiers each of which depends on the coefficient of risk aversion, ν (see Shalit and Yitzhaki, 1989). Investors have the choice to opt for the portfolios that best suit their aversion to risk. Asset allocation using *MEG* is somewhat similar to *MV* portfolio optimization when short sales are allowed and when return distributions are exchangeable.[2] In that case, standard *MV* algorithms can be used for *MEG* as done by Shalit and Yitzhaki (2005). We now write Problem (5) in matrix form:

$$\begin{aligned} &Min\, \Gamma_p(\nu) \\ \text{s.t.} \quad &E(p) = \boldsymbol{w}'\boldsymbol{\mu} \\ &1 = \boldsymbol{w}'\boldsymbol{l} \\ &\boldsymbol{w} \geq 0 \quad . \end{aligned} \tag{6}$$

where, $\boldsymbol{\mu}$ is the vector of assets' mean returns $\boldsymbol{w}$ is the vector of portfolio weights and $\boldsymbol{l}$ is a vector of ones. Problem(6), although similar in structure to the *MV* optimization problem, is much more complex than the *MV* problem because the extended Gini of a portfolio cannot be derived as a simple function of the probability distribution statistics of the assets. Furthermore, when short sales are not allowed or when distributions are not exchangeable specific optimization programming is needed to

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[2] A set of random variables is exchangeable if for every permutation of the n subscripts, the joint distributions of (x_{j1}, ... , x_{jn}) are identical (Stuart and Ord, 1994). The multivariate normal is an example of an exchangeable distribution up to a linear transformation.

solve the portfolio allocation problem. These techniques are presented in the next section.

3. The Optimization Model

We construct *MEG* efficient portfolios by developing an algorithm based on numerical optimization. In practice, calculating the Gini of a random variable can be done by either averaging the absolute differences between all observations pairs or estimating the *CDF* by the rank function and applying it in the covariance formula. In general, because Gini derivatives are discontinuous, researchers and analysts are refrained from using analytical solutions to construct optimal *MEG* portfolios as gradient-type optimizations approaches fail.

Therefore, linear programming (*LP*) techniques have been proposed by Okunev (1991) to solve for MG efficient portfolios that minimize the Gini expressed by the expected value of absolute differences subject to a required portfolio mean constraint and a portfolio constraint. The absolute value formulation of the Gini causes the primal *LP* problem to contain a number of constraints as large as the number of observations. By moving the primal problem to the dual *LP* program, the number of constraints is reduced to the number of assets and the frontier can be easily solved for a small number of securities (50) but seems intractable for large portfolios. Also, Okunev's *LP* does not offer a solution when minimizing the extended Gini portfolios.

Recently, a simple solution to the mean-Gini portfolio optimization problem was obtained by Cheung, Kwan, and Miu (2005) using a standard *Excel* spreadsheet technique. Here, we implement the approach to *MEG* for a smaller number of securities and allow for risk aversion differentiation. This spreadsheet technique is limited to smaller number of securities and a sizable number of observations. Our challenge was to use an advanced software package such as *Mathematica* to develop a reliable numerical optimization to find efficient portfolios that minimize the extended Gini subject to required expected returns for a large number of securities with and without short sales. We now present the construction of *MEG* efficient portfolios using the two techniques:

3.1 MEG Optimization Using Excel

The *MG* portfolio optimization technique was developed by Cheung et al (2005). The approach is as follows: First, for a given set of portfolio weights, w_i, $i=1,\ldots, n$ compute the portfolio returns as $p=\sum_{i=1}^{N} w_i r_i$. Hence, the portfolio's Gini is calculated as

$$\Gamma_p = 2\,\mathrm{cov}[p, F_p(p)]. \tag{7}$$

To estimate the *CDF*, rank the portfolio returns sorted in ascending order and divide the ordinal rank by the number of observations. Let us consider an example of 10 stocks and 52 returns. In the *Excel* spreadsheet, write down the individual stock returns in columns `A` to `J` and rows `1` to `52`. The portfolio weights are stored in row `54` as `A54:J54`. Now, calculate the portfolio returns in cells `K1:K52` using the function `=SUMPRODUCT(A1:J1,A$54:J$54)`. With the `RANK` function and the `COUNT` function create the *CDF* in column `L` as `=RANK(K1:K52,K$1,K$52,1)/COUNT(K1:K52)`. The portfolio Gini is calculated by: `=2*COVAR(K1:K52,L1:L52)`. The main feature of this process is rooted in the spreadsheet procedure that instantaneously and simultaneously updates the *CDF* of the portfolio whenever portfolio returns are computed for a specific set of weights. This is why *Excel* and the `RANK` function are so successful in providing an optimization solution for the mean-Gini frontier.

Now, we can adapt the *MG* technique and construct the *MEG* frontier. For a specific coefficient of risk-aversion ν, the extended Gini of a portfolio is expressed as:

$$\Gamma_p(\nu) = -\nu\,\mathrm{cov}\{p,[1-F_p(p)]^{\nu-1}\}. \tag{8}$$

The formula is calculated by `=- A55*COVAR(K1:K52,(1-L1:L52)^(A55-1))` where ν is stored in cell `A55`. MEG optimization is achieved using the *Excel Solver* by minimizing the portfolio extended Gini subject to the required mean return, the portfolio weights constraint and whether short sales are allowed or not.

3.2 The Mathematica MEG Optimization Technique

We use *Mathematica* to write a general numerical algorithm that minimizes the extended Gini of a portfolio for a given set of risky assets. The algorithm is structured into a main routine that calls on specific subroutines responsible for tasks such as finding the minimum extended Gini or constructing the efficient frontiers for a set of risk aversion parameters ν. The main routine loads the data including asset names and asset returns. Then, the analyst is asked to specify several parameters. The first parameter determines whether the required expected return on the portfolio is set to specific values (*RestrictRange* =1) or the unconstrained minimum (extended) Gini is needed to be computed (*RestrictRange=0*). In the case of specific required returns, one is requested to specify the lower bound (*MPR*), the step size (*RPS*), and the number of steps (*NumberofSteps*) for these returns. A set of required returns is established for which the efficiency frontier will be computed. Afterwards, the analyst is asked to choose what risk measures will be used: the Gini (*MethodRange=0*), the extended Gini (*MethodRange=1*), both risk measures (*MethodRange=2*), or else many extended Ginis for various risk-aversions ν (*MethodRange=3*). For the latter, a 3-dimentional efficient frontier in the space spanned by expected return, the extended Gini, and the risk-aversion parameter {E(p), Γ_ν(p), ν }may be constructed.

Conditional on choosing the extended Gini optimization, the analyst is asked for a specific risk-aversion ν (*RiskAversionParameter*) or for the information required to construct a set of risk aversion parameters: the lower limit (*RiskAversionStart*), the step size (*RiskAversionSize*) and the number of steps (*RiskAversionSteps*). She is also asked whether short sales are allowed (*NoShortSales=0*) or not (*NoShortSales =1*).

Following this parameterization, the program loads the two subroutines *OptPortfolioGini* and *OptPortfolioExtGini* the tasks of which are presented in detail below. The starting minimal portfolio required return is specified. Conditional on the values for *MethodRange* and *RestrictRange* the routine takes different paths. If the unconstrained (extended) Gini is chosen the program computes either the minimum Gini using *OptPortfolioGini* or the minimum extended Gini using *OptPortfolioExtGini* and reports these values in the object *ResultVector*. If the decision is to compute an efficient frontier, the program enters, depending on the value chosen for *MethodRange*, into different loops. If only one risk aversion coefficient is implicitly or explicitly specified, for each step of the loop *OptPortfolioGini*, *OptPortfolioExtGini* or both are computed, the results are added to

ResultVector (*ResultbVector* or both of them) and the required minimal portfolio return is increased by *RPS*.

Finally, if *MethodRange* indicates several values for the risk-aversion coefficient, the routine enters the alternative branch of two nested loops following the risk aversion coefficient defined by *RiskAversionStart*. The outer loop runs through the various risk aversion coefficients and the inner loop runs through the various required portfolio returns. After finishing the inner loop, the risk aversion is increased by the value of *RiskAversionSize*, the results *are* added to *ResultArray*, and the routine starts the outer loop. This procedure is repeated for *RiskAversionSteps* times.

The subroutine *OptPortfolioExtGini* minimizes the extended Gini of the portfolio given a required portfolio return for a specified risk aversion parameter. The extended Gini is computed in the subroutine *PortfolioExtGini*. The minimization method used is a linear programming technique based on a simplex algorithm. Similarly, *OptPortfolioGini* computes the portfolio by minimizing the Gini for a required return and an input matrix of asset returns. The portfolio Gini is calculated in the subroutine *PortfolioGini.*

The flowchart presented in Figure 3 visualizes the logic structure of the software package explained above.

4. Empirical Analysis and Results

We use as data the monthly returns of 100 most valued traded stocks on the US financial markets from March 1992 until June 2007. Most of these firms appear in the S&P 100 index. The 183 returns were calculated from monthly close price adjusted for dividends and splits downloaded from finance.yahoo.com. The summary statistics (mean, standard deviation and Gini mean difference are presented in Table 1 together with the Jarque-Bera test statistic for the normality of returns. For most of the firms normality is rejected, justifying the use of the Gini as an appropriate risk measure to obtain SSD portfolios.

The first stage consists of constructing the MEG efficient frontier for a variety of ν using the Excel routine. The efficient portfolios are calculated when short sales are not allowed for ν = 2, 3, 4, 6, 8, 10, 15, 20, 40, 60, 80, 100. The results are shown in Figure 2. In general, the various portfolio frontiers seem almost the same in shape, concavity, and mean return corresponding to the minimum extended Gini. But still a sizeable change in the trade-off between the mean return of the portfolio and the risk expressed in (extended) Gini can be observed. A reduction of the mean return from 0.023 to 0.021 yields a risk improvement of 0.02 for $\nu = 2$, but reduces risk by almost 0.1 for ν=100. Hence, the trade-off worsens considerably, reflecting the higher risk aversion of the investor. The holdings of the minimum extended Gini portfolios are presented in Table 2. The results display similar patterns with respect to the efficient frontiers. Some sizable differences between the various portfolios can be observed: the portfolios with higher risk aversion ν engage in stronger diversification and the optimal portfolios with lower ν rely on a relative low number of assets.

The second stage of the optimization is performed by using the Mathematica software. It generates efficient portfolios frontiers by varying the mean return, the extended Gini, and the risk aversion parameter. The resulting 3-dimentional efficient frontiers surfaces are displayed in Figure 4. In here, a rising risk aversion parameter is accompanied by an increase in the risk compensation needed for a given reduction in expected return. Furthermore, the figure illustrates that the speed of the trade-off change does not follow a monotone pattern, but rather appears as a volatile process. On the margin, it turns out that not even the change in the trade-off is a strictly monotone function in the risk aversion parameter.

This rather unexpected result cannot only be explained by the non-continuous adjustments in the optimal weights of the remaining assets whenever security is added or removed from the optimal portfolio, but also from trade-offs inherent in the risk aversion parameter itself. This argument can be explained using the elasticity of the extended Gini with respect to the risk aversion parameter ν. From Equation (3) we obtain the derivative of the extended Gini. Therefore, the elasticity w.r.t. ν is:

$$\frac{\partial \Gamma_x(\nu)}{\partial \nu}\frac{\nu}{\Gamma_x} = \frac{2\nu - 1}{\nu - 1} + \nu \int_0^1 \ln(1-\varphi)d\varphi \tag{9}$$

The ratio $\frac{2\nu - 1}{(\nu - 1)}$ and the second term of Equation (9) create trade-offs that can lead to non-monotony. In particular, this ratio exhibits singularity when $\nu \to 1$ or $\nu \to 0$ implying that near these values the extended Gini elasticity is non-monotonous. This feature adds on top of the trade-off between risk bearing and diversification that can raise the required mean returns.

5. Conclusions

We have presented a new approach to construct MEG portfolios by inserting the coefficient of risk-aversion into the optimization program. Hence the results show a three dimensional frontier where the risk-aversion coefficient can be chosen to enhance the risk inherent in the portfolios. Not only the results deliver stochastic dominant portfolios but they allow the analyst to offer a variety of alternatives for risk-averse investors.

In addition, the paper provides some innovations of a more technical nature. These include a Mathematica algorithm consisting of several interdependent Mathematica packages and a notebook that allows for the efficient computation of hulls by varying portfolios for a predefined set of assets. The size of this set is only restricted by the computational resources available. In any case, these resources can be improved by using the inherent parallel computing capacities of Mathematica. In this way, the number of elements within the set of assets serving as the main input to the algorithm can be almost unlimited.

It is exactly this computational power that allows also for a high degree of flexibility in the design of the objective function within the process of the portfolio optimization (5). Hence, future extensions of this research could include taking into account higher moments and/or co-moments, additional parameters besides the risk-aversion parameter and even a more general functional. Another potentially interesting extension would be to include an objective function that not only reflects the trade-off between return and risk, but also uses a measure of financial stability so that minor shocks on the exogenous parameters and variables would not result in major portfolio restructuring. Building on this argument, integrating transaction costs into the objective function could potentially convey interesting results as well.

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Table 1: Summary Statistics for 100 stocks (Monthly Returns March 1992 – June 2007)

Firm	Mean	Std Dev	Gini	JB Stat	Firm	Mean	Std Dev	Gini	JB Stat
AA	1.38%	9.30%	5.25%	131.82	HPQ	1.64%	10.88%	6.14%	6.76
ABT	1.09%	5.96%	3.36%	6.26	IFF	0.78%	6.45%	3.64%	36.34
AAPL	2.31%	14.75%	8.32%	8.67	IBM	1.40%	9.13%	5.15%	15.33
AEP	0.86%	6.07%	3.43%	9.82	INTC	2.23%	12.07%	6.81%	7.71
AES	2.21%	16.37%	9.23%	263.03	IP	0.61%	7.68%	4.33%	21.22
AIG	1.37%	6.46%	3.64%	22.71	JNJ	1.22%	6.05%	3.41%	0.42
AMGN	1.56%	9.91%	5.59%	55.14	JPM	1.50%	8.95%	5.05%	32.40
AVP	1.75%	9.22%	5.20%	292.58	KO	0.86%	6.45%	3.64%	14.84
AXP	1.66%	6.87%	3.88%	49.9	LTD	1.21%	9.84%	5.55%	11.00
BA	1.26%	7.82%	4.41%	37.76	MCD	1.23%	6.82%	3.85%	10.67
BAC	1.34%	6.95%	3.92%	28.43	MDT	1.64%	7.02%	3.96%	5.75
BAX	1.20%	7.51%	4.24%	73.44	MER	2.06%	9.57%	5.40%	10.53
BHI	1.58%	9.82%	5.54%	5.47	MMM	1.15%	5.94%	3.35%	29.54
BMY	0.83%	6.82%	3.85%	28.73	MO	1.46%	8.33%	4.70%	35.37
BNI	1.38%	6.87%	3.87%	2.58	MRK	0.94%	7.86%	4.44%	1.51
BUD	1.02%	4.86%	2.74%	0.23	MSFT	1.95%	10.18%	5.74%	30.74
BDK	1.25%	8.60%	4.85%	3.87	MAY	0.67%	11.94%	6.74%	1247.2
BC	1.07%	9.57%	5.40%	69.95	MEE	1.46%	12.63%	7.13%	41.94
C	2.06%	8.39%	4.74%		NSC	1.18%	7.99%	4.51%	13.19
CAT	1.92%	8.36%	4.72%	31.22	NSM	2.27%	16.32%	9.21%	8.23
CCU	2.54%	10.42%	5.88%	18.58	NT	1.30%	19.55%	11.03%	953.11
CI	1.83%	8.92%	5.03%	222.53	ORCL	3.20%	14.23%	8.03%	77.35
CL	1.41%	7.15%	4.04%	94.87	OMX	0.96%	8.77%	4.95%	1.96
CMCSA	1.66%	9.39%	5.30%	7.51	OXY	1.58%	7.65%	4.32%	14.84
COP	1.51%	6.84%	3.86%	8.4	PEP	1.10%	6.17%	3.48%	74.30
CPB	0.97%	6.54%	3.69%	2.8	PFE	1.21%	6.82%	3.85%	1.57
CSC	1.41%	10.07%	5.68%	134.2	PG	1.22%	6.17%	3.48%	389.57
CSCO	2.89%	12.11%	6.83%	5.12	RF	0.91%	5.72%	3.23%	9.66
CVS	1.30%	7.84%	4.42%	28.54	ROK	2.36%	9.30%	5.24%	83.38
CVX	1.35%	5.56%	3.14%	14.86	RTN	0.92%	8.48%	4.78%	162.61
CEN	1.52%	8.76%	4.94%	5.47	RSH	1.56%	11.38%	6.42%	0.98
DD	0.89%	6.71%	3.79%	0.51	S	1.33%	9.42%	5.31%	65.71
DELL	3.42%	14.79%	8.34%	5.98	SLB	0.74%	7.05%	3.98%	37.35
DIS	0.89%	7.50%	4.23%	13.08	SO	1.58%	5.42%	3.06%	27.79
DOW	1.03%	7.59%	4.28%	131.58	T	1.12%	7.31%	4.12%	11.74
EK	0.57%	8.45%	4.77%	42.66	TEK	1.91%	12.69%	7.16%	54.91
EMC	3.40%	14.79%	8.35%	1.83	TGT	1.79%	7.93%	4.48%	2.58
EP	1.28%	10.96%	6.18%	150.36	TWX	4.03%	16.08%	9.07%	88.95
ETR	1.65%	6.42%	3.62%	23.56	TXN	2.50%	12.89%	7.27%	10.17
EXC	1.55%	6.85%	3.86%	50.88	TYC	1.66%	9.86%	5.56%	207.48
F	0.86%	9.56%	5.39%	20.96	USB	1.42%	7.43%	4.19%	131.57
FDX	1.59%	8.45%	4.77%	21.41	UTX	1.73%	7.04%	3.97%	125.09
GD	2.70%	9.10%	5.14%	1951.85	VZ	0.98%	7.29%	4.11%	152.00
GE	1.36%	6.08%	3.43%	3.49	WB	0.94%	7.24%	4.08%	47.55
GM	0.88%	9.61%	5.42%	0.49	WFC	1.56%	6.59%	3.72%	19.20
HAL	1.87%	10.97%	6.19%	9.69	WMB	2.18%	13.01%	7.34%	267.65
HD	1.30%	8.07%	4.55%	4.13	WMT	1.02%	7.14%	4.03%	4.43
HET	2.25%	10.74%	6.06%	112.68	WY	1.14%	7.41%	4.18%	0.49
HNZ	0.93%	5.58%	3.15%	0.79	XRX	1.18%	12.79%	7.21%	505.79
HON	1.40%	9.16%	5.17%	339.23	XOM	1.35%	4.74%	2.68%	56.74

Table 2: Holdings of Minimum Extended Gini Portfolios for various ν, showing only the non-zero positions

Firms	ν=2	ν=3	ν=4	ν=6	ν=8	ν=10	ν=15	ν=20	ν=40	ν=60	ν=80	ν=100
ABT	2.99%	4.69%	5.48%	5.58%	5.64%	5.28%	5.14%	4.63%	5.07%	5.07%	5.07%	5.07%
AAPL	0.23%	0.61%	1.01%	2.10%	3.39%	4.84%	5.49%	6.00%	7.11%	7.12%	7.12%	7.12%
AMGN	1.40%	1.21%	0.97%	0.82%	0.75%	2.60%	2.53%	2.94%	2.16%	2.16%	2.16%	2.16%
BNI	1.68%	1.76%	1.72%	1.51%	1.22%	0.10%	0.10%	0.09%	0.08%	0.08%	0.08%	0.08%
BUD	17.50%	15.20%	13.85%	12.24%	11.55%	11.79%	11.71%	11.24%	13.16%	13.18%	13.18%	13.18%
COP	0.00%	0.00%	0.09%	0.09%	0.09%	0.13%	0.14%	0.14%	0.13%	0.13%	0.13%	0.13%
CPB	0.00%	0.00%	0.00%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%
CVX	8.91%	8.38%	7.21%	5.72%	4.87%	4.26%	3.68%	2.15%	0.69%	0.69%	0.69%	0.69%
CEN	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.50%	0.50%	0.50%	0.50%
DOW	2.22%	2.55%	2.44%	2.25%	2.65%	3.87%	3.61%	2.51%	1.25%	1.25%	1.25%	1.25%
EK	2.96%	3.63%	3.76%	4.15%	4.38%	3.39%	3.57%	4.03%	5.05%	5.06%	5.06%	5.06%
EXC	5.29%	4.34%	3.84%	3.46%	3.50%	3.13%	1.84%	1.18%	1.19%	1.19%	1.19%	1.19%
FDX	0.70%	0.76%	0.79%	0.78%	0.69%	0.34%	0.33%	0.32%	0.31%	0.31%	0.31%	0.31%
GD	2.02%	1.18%	0.78%	0.51%	0.46%	0.84%	1.12%	2.74%	7.05%	7.05%	7.05%	7.05%
HET	0.35%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.20%	0.20%	0.20%	0.20%	0.20%
HNZ	8.00%	8.10%	8.47%	7.82%	8.16%	8.05%	8.54%	8.08%	4.02%	4.02%	4.02%	4.02%
LTD	0.00%	0.63%	0.74%	0.91%	1.16%	3.87%	4.17%	5.06%	5.42%	5.41%	5.41%	5.41%
MDT	2.01%	2.26%	2.25%	1.72%	1.34%	0.00%	0.00%	0.00%	0.60%	0.60%	0.60%	0.60%
MER	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.06%	0.06%	0.06%	0.06%
MMM	4.27%	3.65%	2.71%	2.12%	1.80%	0.21%	0.20%	0.16%	0.14%	0.14%	0.14%	0.14%
MO	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.14%	0.11%	0.11%	0.11%	0.11%
MSFT	3.61%	2.61%	2.39%	2.42%	2.22%	1.69%	1.33%	1.10%	0.59%	0.59%	0.59%	0.59%
NSM	1.05%	0.87%	0.87%	0.91%	0.70%	0.14%	0.15%	0.11%	0.09%	0.09%	0.09%	0.09%
ORCL	1.53%	2.27%	2.59%	3.60%	3.69%	4.12%	3.92%	3.55%	3.91%	3.90%	3.90%	3.90%
PG	0.15%	0.11%	0.10%	0.10%	0.10%	0.17%	0.18%	0.13%	0.11%	0.11%	0.11%	0.11%
RSH	3.62%	4.31%	4.40%	3.91%	3.26%	2.08%	1.36%	0.65%	0.28%	0.28%	0.28%	0.28%
S	0.00%	0.68%	0.86%	0.91%	0.91%	0.45%	0.44%	0.40%	0.30%	0.30%	0.30%	0.30%
SLB	0.00%	0.00%	0.00%	0.04%	0.04%	0.04%	0.04%	0.05%	0.05%	0.05%	0.05%	0.05%
SLE	0.00%	0.00%	0.00%	0.84%	0.88%	1.44%	1.66%	1.77%	1.57%	1.57%	1.57%	1.57%
SO	19.66%	23.58%	25.70%	28.02%	29.11%	31.13%	32.48%	33.32%	31.94%	31.86%	31.86%	31.86%
T	0.00%	0.00%	0.00%	1.27%	1.44%	2.79%	2.82%	2.78%	0.70%	0.70%	0.70%	0.70%
TEK	2.75%	2.44%	2.60%	2.91%	3.12%	3.11%	3.31%	4.32%	5.74%	5.73%	5.73%	5.73%
Total	92.91%	95.80%	95.62%	96.73%	97.13%	99.88%	99.88%	99.81%	99.61%	99.54%	99.54%	99.54%

Figure 1: The Absolute Lorenz Curve

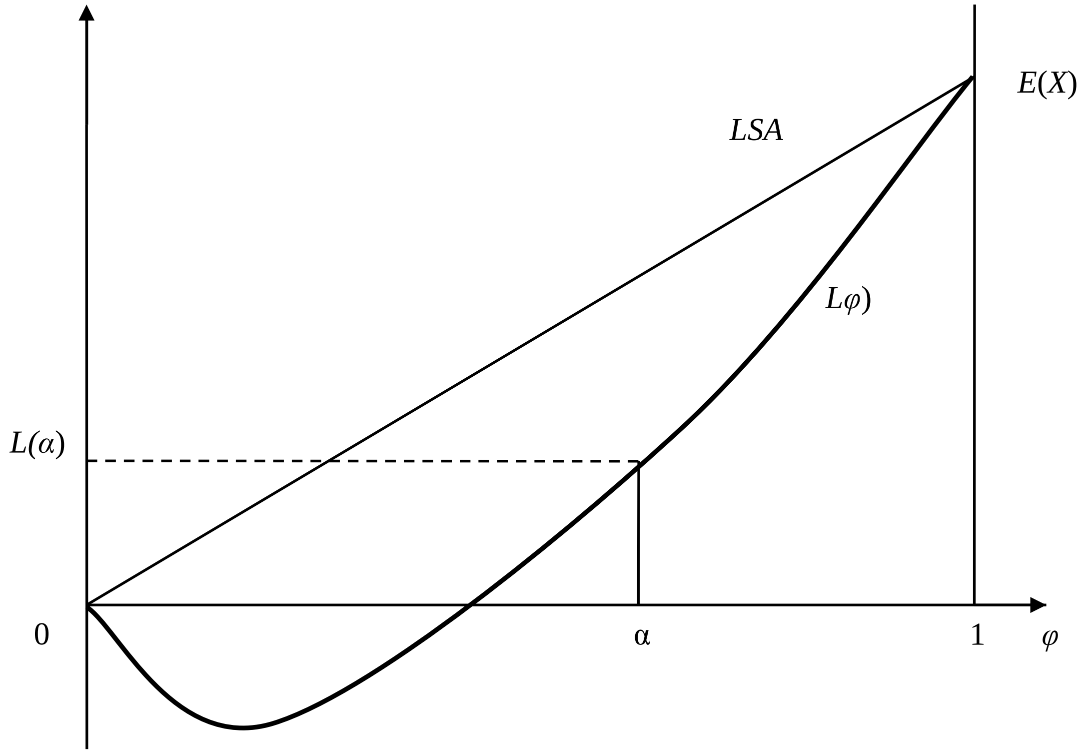

Figure 2: Efficient Frontiers for Various νs

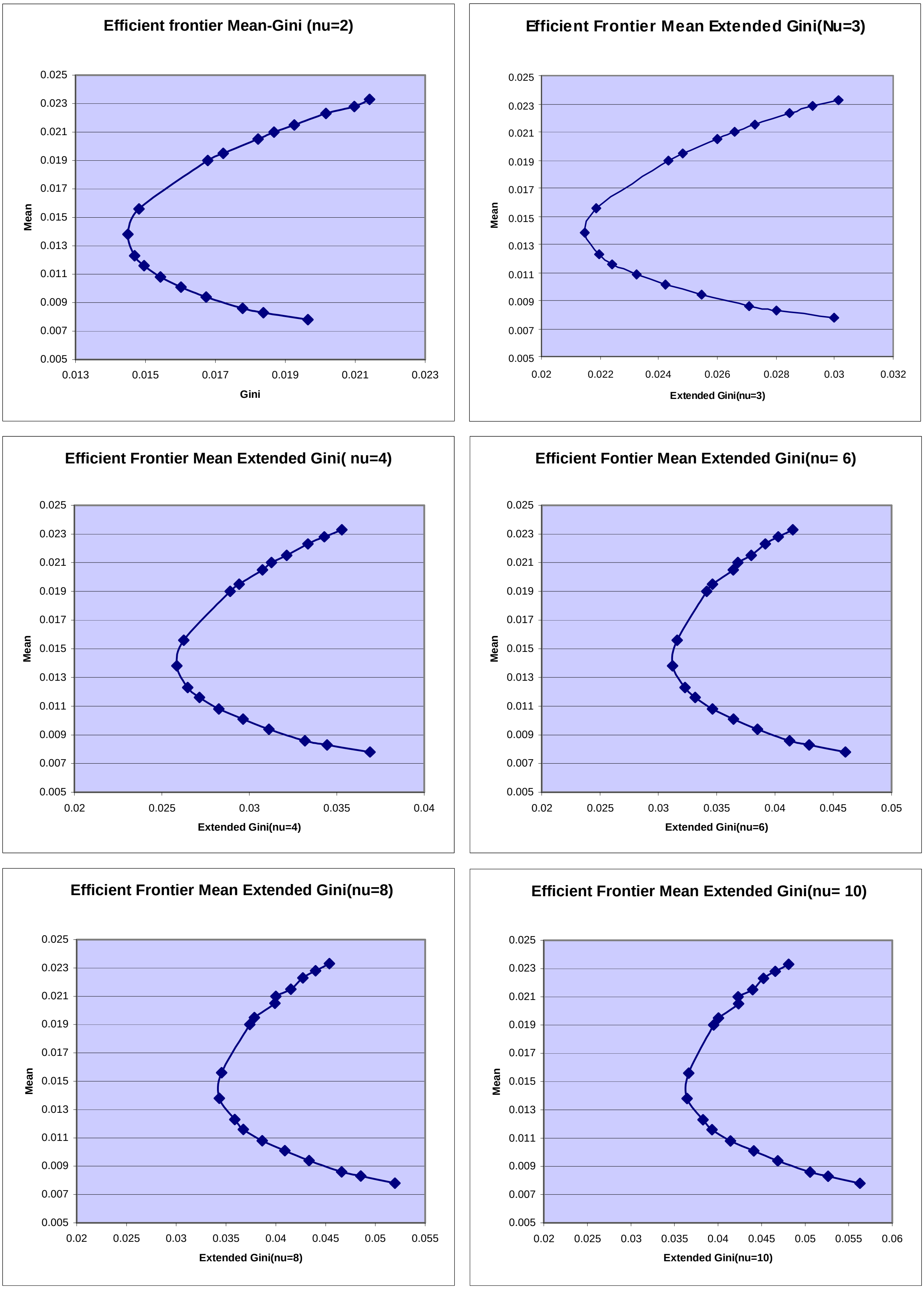

Figure 2: cont.

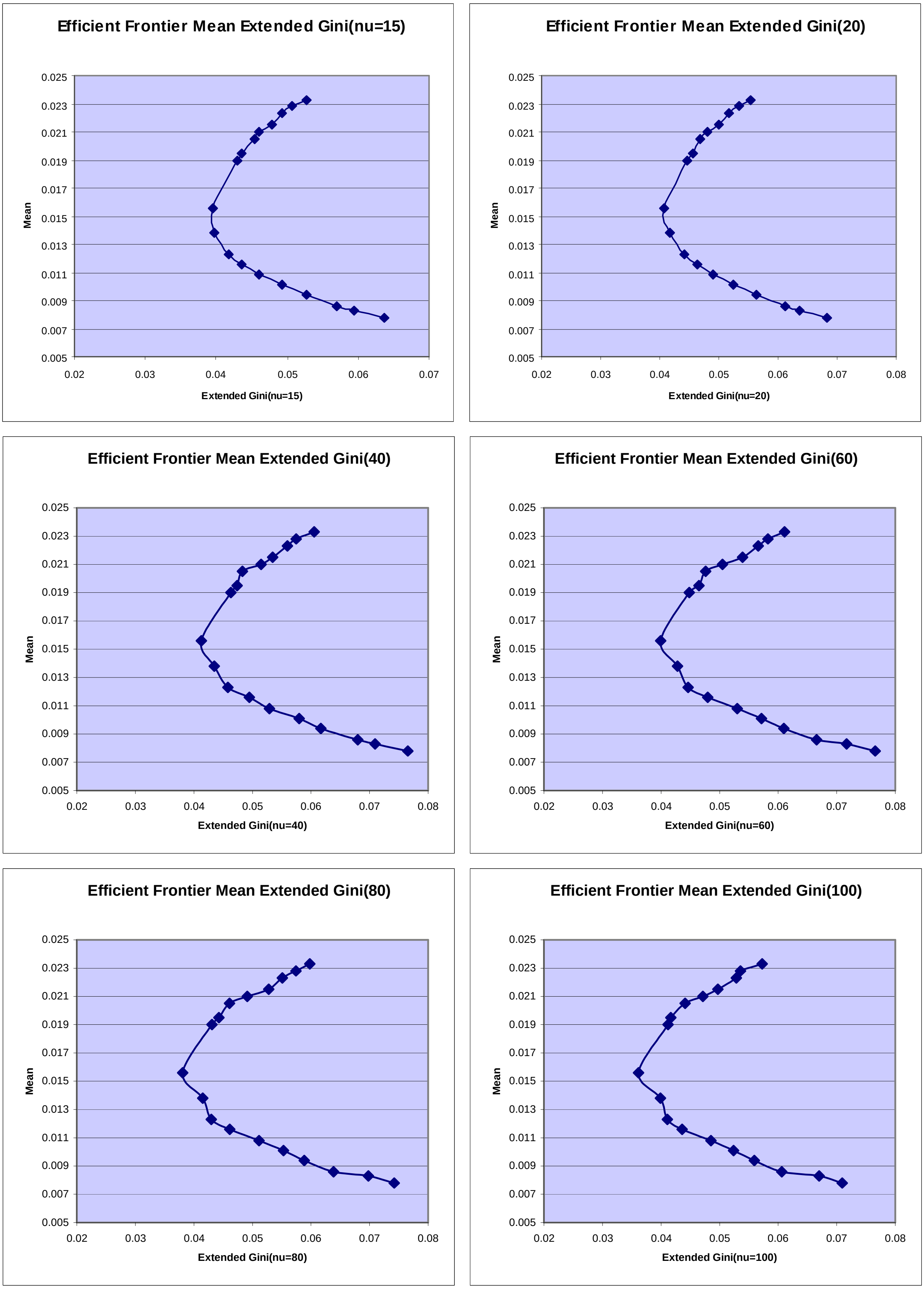

Figure 3: Flowchart for Mathematica Package

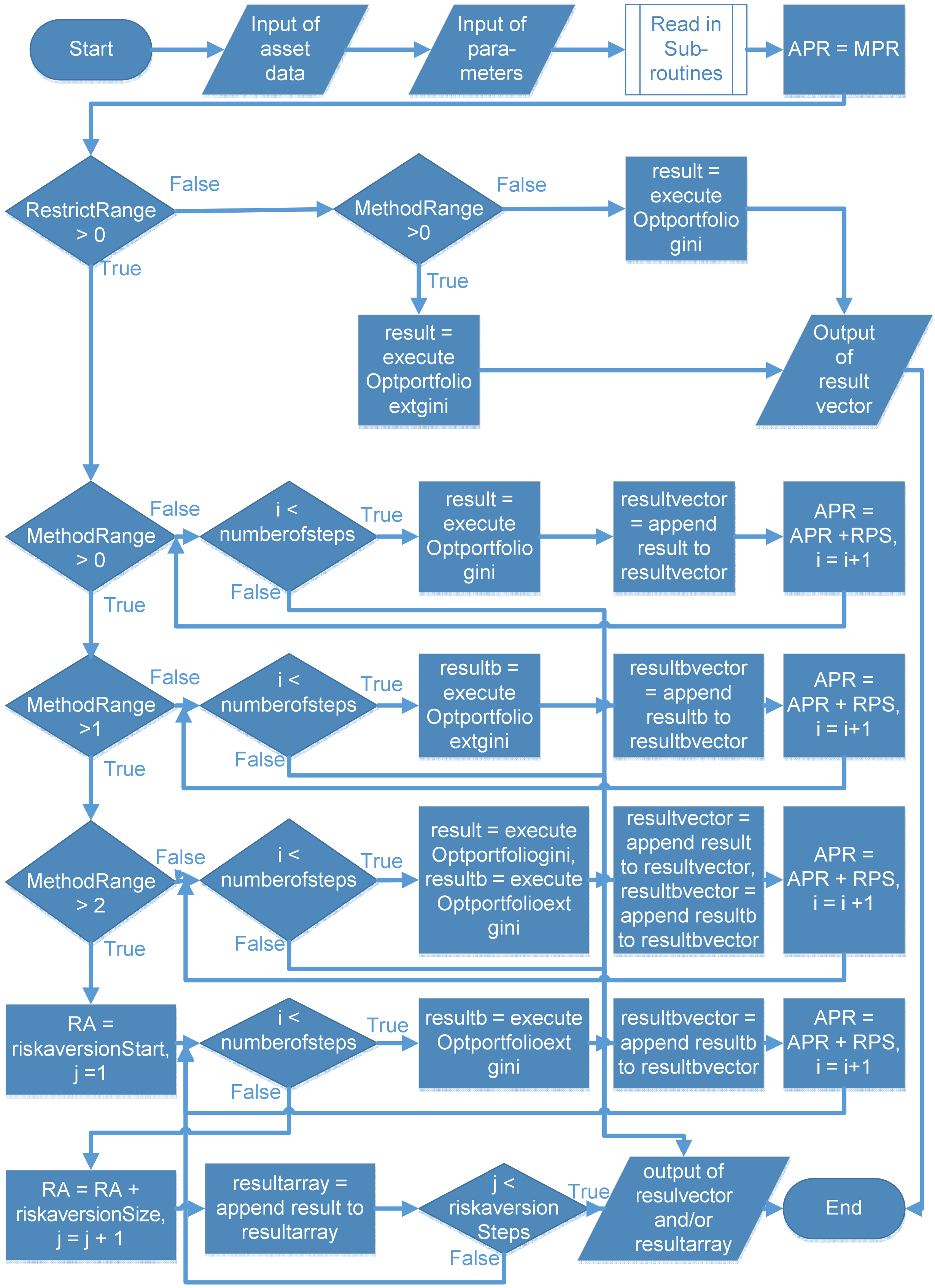

Figure 4: 3-D Efficiency Frontier Mean Extended Gini