Desirability relations in Savage's model of decision making*

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Abstract

The subjective probability of a decision maker is a *numerical* representation of a *qualitative probability* which is a binary relation on events that satisfies certain axioms. We show that a similar relation between numerical measures and qualitative relations on events exists also in Savage's model. A decision maker in this model is equipped with a unique pair of probability on the state space and cardinal utility on consequences, which represents her preferences on acts. We show that the numerical probability-utility pair is a representation of a family of *desirability relations* on events that satisfy certain axioms. We first present axioms on a desirability relation defined in the *interim* stage, that is, after an act has been chosen. These axioms guarantee that the desirability relation is represented by a probability-utility pair by taking for each event conditional expected utility. We characterize the set of representing pairs by measuring the *optimism* of probabilities on consequences and the *content* of utility functions. We next present axioms on the way desirability relations are associated *ex ante* with various acts. These axioms determine the unique probability-utility pair in Savage's model.

1 Introduction

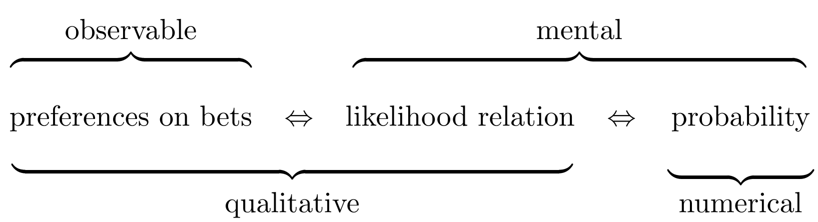
1.1 Mental vs. observable, numerical vs. qualitative

The early works on decision theory introduced *subjective probability*. Ramsey (1926) and de Finetti (1931, 1937), the champions of this idea, perceived

*Acknowledgements will be added.

it as an individual's *mental* attitude. It has the great merit of being a *numerical* measure, which makes it very useful in both theory and applications. But, being a mental entity it is hidden and immune to observation. Preferences on bets, in contrast, induce *observable* behavior, and they are *qualitative*. The idea that the mental numerical probability is equivalent to the observable qualitative preference on bets was not fully formalized but it permeated the decision theory literature at the time.

Despite the usefulness of probability, it is suspicious as a mental entity. People seldom make precise probabilistic assessments like "the probability of prices falling next week is 0.756". Thus, we can say that even if minds have, perhaps unconsciously, a numerical probability measure it is rarely reported. This motivated an attempt by de Finetti (1931) to replace numerical probability by an equivalent *qualitative* mental entity. Such an entity was called *qualitative probability*, or *likelihood relation* on events. Like numerical probability it is not directly observable, but unlike it, it is routinely reported. Thus, the theory consists of three, supposedly equivalent, parts: observable qualitative preferences on bets, a mental numerical probability, and an intermediate part, qualitative mental preferences of likelihood relation, as depicted in the following diagram.



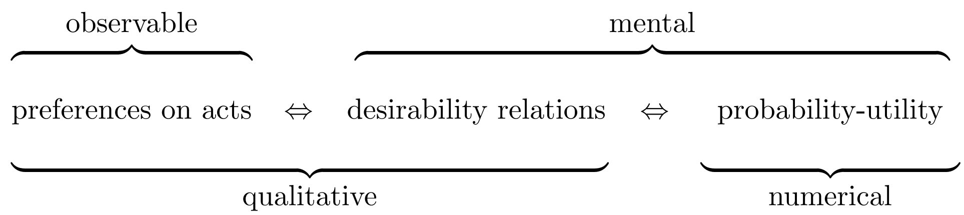
One may hold a reductionist view that the likelihood relation is nothing more than preferences on bets in disguise. We adopt here the view that it is a mental attitude that exists independently of bets, and perhaps conceptually prior to them. People seldom bet but frequently report on some event being more likely than others, a notion of which they have an intuitive grasp. One can think of the binary likelihood relation as making the monadic relation of being true fuzzier. Some events are 'truer' than others, that is, more likely.

Utility is the other major subjective component of decision theory, and like probability it is mental and numerical. von Neumann and Morgenstern (1953) showed that a preference relation over lotteries on *prizes* is the qualitative observable counterpart of a utility function on these lotteries.

The marriage of utility and probability in one theory of decision making was accomplished by Savage (1954). In his theory the mental numerical part consists of a pair, probability on a *state space* and utility over *consequences*.

The qualitative observable part is preferences on acts, where the consequence of an act depends on the state of the world.

If reports on the numerical value of a probability are rare, reports on the numerical value of utility are even rarer. Thus, finding a third equivalent part of the theory which will be a qualitative mental relation on events is even more urgent in Savage's theory than in the early theory on bets. The purpose of this paper is to introduce such a third part into Savage's theory. This binary relation on events is called *desirability*, and it is an extension of the monadic relation " I wish that..." or "I desire that...". Thus, desire is now comparable: one event is more desirable than another. The following diagram depicts in broad strokes the structure we intend to present.



Binary relations on propositions that capture both probability and utility were studied by Jeffrey (1965) and Bolker (1967). But their model did not include consequences and *a fortiori* acts, and utility was defined on propositions. Thus, their work was incompatible with decision theory as formulated by Savage. Moreover, the lack of acts defined on the same state space made it impossible, in their framework, to determine a *unique* probability-utility pair.

1.2 *Ex ante* and interim in Savage's model

At first glance a desirability relation seems to be at odds with Savage's model. In order to say that one event is more desirable than another event, these events should be related to consequences, which are the carriers of utility in Savage's model. However, events in this model, that is, subsets of the states of the world, are independent of consequences. A state is only a *partial* description of the world that includes all that is needed to determine the consequence of an act, but of course does not describe the consequence itself. Therefore, in such a model a relation between events cannot be related to utility.

For a relation on events to capture utility of consequences, we should have a *comprehensive* state space each state of which is a *full* description of the world that include also a specification of a consequence. We are not aware of such a model in the literature of decision theory. But in the last three decades comprehensive state spaces play an important role in game theory. The reason for this is that in order to analyze the reasoning of players in a game one needs to describe as events their choices and the consequences that follow. Thus, the states should include a specification of the choices and the consequences. This idea was first implemented in Aumann (1987) where he claimed that the use of comprehensive state spaces was the main novelty of the model in that paper.

> The chief innovation in our model is that it does away with the dichotomy usually perceived between uncertainty about acts of nature and of personal players. [...] In our model [...] the decision taken by each decision maker is part of the description of the state of the world. (Aumann, 1987)

The reason why comprehensive state spaces are required here is the same as in the game theoretic setup. In order to compare desirability of events we need events to be related to consequences, which requires in turn that states specify consequences.

Comprehensive state space can be easily identified within Savage's framework. We note that the setup in Savage (1954) is an *ex ante* model where the states describe all the relevant facts *prior* to the decision maker choosing her act. Thus, states do not specify consequences at this stage. But after the decision maker has made up her mind and has chosen an act we are facing an *interim* stage. At this stage, the uncertainty of the agent about the state of world remains the same, but now a state of the world *does* specify a consequence, the one associated with the state by the chosen act. Thus, in the interim stage when the act is fixed, the state space becomes comprehensive which opens the door to defining a desirability relation on events.

1.3 An outline of the main results

Given a probability on a state space, a utility on consequences, and an act, each state is associated with a consequence and hence with utility. Consider a *desirability relation* on events. We say that the relation is *represented* by a probability-utility pair, if event E is more desirable than event F whenever the expected utility given E is larger than the expected utility given F.

Our first result is a representation theorem for desirability relations. We list seven axioms on a desirability relation that are necessary and sufficient for it to be represented by a probability-utility pair.

We call the event that a certain consequence holds a *consequence event*. We show that for the probability-utility pairs in the representation theorem, the conditional probabilities given consequence events are uniquely determined. However, the probabilities of the consequence events themselves are not unique. These probabilities are ordered by *optimism*. For each probability there exists a unique cardinal utility. The corresponding utilities are ordered by *content*.[1]

We next consider a family of acts and for each act in this family a single desirability relation. We present five axioms on the family of acts and the associated desirability relation. We show that these axioms guarantee the existence of a unique probability-utility pair that represents all the desirability relations associated with the acts in the family.

1.4 Literature survey

Jeffrey (1965) introduced *desirability* as a function defined on the same domain as probability, which in his model, is a set of *propositions*. The novelty in his model is the departure from the old idea of consequences and acts. Translating his definitions to Kolmogorov's set theoretic model of probability, desirability is the quotient of a signed measure on the state space and a probability on this space. Under the right technical assumptions, the signed measure is determined by its Radon Nykodim derivative with respect to the probability. The desirability of an event, then, is the conditional expectation of this derivative.

Bolker (1967) considered a binary relation on propositions, which was not named, and axioms on this relation that guarantee that it can be represented by a desirability measure. He comments on the difference between Jeffrey's model and Savage's model: "The states must be unambiguously described. By so doing we blur the often useful distinctions among acts, consequences and events" (Bolker, 1967, foonote 7). This lost distinction is reinstated here where we use Savage's model in which consequences and acts are the main features. Within Savage's model we distinguish between the *ex ante* stage, before the act is chosen, and the interim stage after it is chosen. For a given interim stage, we consider a binary relation, which we call a *desirability relation*. We present axioms, some of which are similar to those used by Bolker (1967), that guarantee that the relation is represented by a probability on the state space and utility on consequences. We then turn to the *ex ante* stage and consider desirability relations associated with various

[1] A detailed example is presented in Section 5.

acts. We present axioms concerning this association that guarantee that all the desirability relations can be represented by a unique pair of probability and cardinal utility.

In addition to the essential difference of having consequences and acts in our model, it differs from Bolker's model in other aspects. (1) In Bolker (1967) the relation is defined on the non-zero elements of a complete non-atomic Boolean algebra. This corresponds to quotient space of a measurable space with respect to null events. Thus, null events must be defined prior to the definition of the desirability relation. In our model, like in Savage's, null events are defined in terms of the relation rather than assumed. (2) Bolker assumes that the relation is continuous and derives representing probabilities that are σ-additive. We make no continuity assumption and , like Savage (1954), derive an additive probability.

Bolker (1967) and Jeffrey (1983) describe a linear structure of the set of probability-utility pairs that represent the binary relation in their model, a structure that was suggested to Jeffrey by Kurt Gödel (Jeffrey, 1983, p. 143). The characterization of this set in our model depends on its central feature, the set of consequences. This enables us to demonstrate three features of the non-uniqueness in our model: (1) the conditional probability given a consequence event is uniquely determined; (2) the probabilities of the consequence events are ordered by optimism; and (3) a cardinal utility for a given probability is uniquely determined and the utility gains are ordered by content.

Luce and Krantz (1971) used conditional expected utility to represent a binary relation. However, unlike desirability, the relation they study is not defined on events but on *conditional acts*, namely acts that are not a function on the whole state space but only on an event in this space.

2 The model

Let (Ω, Σ) be a *state space* where Ω is the set of states and Σ is a σ-algebra of events. A finite set $\mathcal{C} = \{c_1, \ldots, c_n\}$ with $n \geq 2$ is the set of *consequences*. An *act* is a measurable function $f \colon \Omega \to \mathcal{C}$ that specifies a consequence in each state.

We refer to (Ω, Σ) as an *ex ante* space because its states do not describe the act of the decision maker, and hence also not the consequence of her act. For a fixed act f we refer to (Ω, Σ, f) as an *interim state space*. Such a space is *comprehensive* in the sense that each state of the world can be thought of as a *full* description of the world, including the consequence at the state specified by f.

Fixing an interim space (Ω, Σ, f), we consider a binary *desirability relation*, $\succsim$, on Σ. We read $E \succsim F$ as 'E is *at least as desirable as* F'. We denote by $\sim$ the symmetric part of $\succsim$. That is, $E \sim F$ when $E \succsim F$ and $F \succsim E$. We read, $E \sim F$ as 'E is *as desirable as* F', or 'E is *similar* to F'. We denote by $\succ$ the asymmetric part of $\succsim$. That is, $E \succ F$ when $E \succsim F$ but not $F \succsim E$. The relation $E \succ F$ is read as 'E is *more desirable than* F'. We consider below Axioms Int 1–Int 7 ('Int' for interim) that desirability relations should satisfy.

2.1 Null events

Events that have no impact on the desirability relation are defined below as null events for this relation. In Savage (1954) null events are those that have no impact on the preference relation on acts. Defining null events in Savage's model requires the notion of *conditional preferences given an event*, which to be well defined requirs axiom P2 to hold. Here, in contrast, where the relation is between events, we define null events prior to imposing any axiom on the relation.

In the definition that follows, we denote by $A\Delta B$, for events A and B, the symmetric difference of the two events.[2]

Definition 1. (Null events) *An event N is* null *for the relation $\succsim$ when for all events E and F, if $E \succsim F$ ($E \not\succsim F$), then also $E' \succsim F'$ ($E' \not\succsim F'$) for any E' and F' that satisfy $(E'\Delta E) \cup (F'\Delta F) \subseteq N$.*

An immediate corollary of the definition is that null events do not affect any of the relations $\succ$, $\sim$, $\not\succ$, and $\not\sim$.

Corollary 1. *If E and F satisfy some of the relations $\succ$, $\sim$, $\not\succ$, and $\not\sim$, N is a null events, and $(E'\Delta E) \cup (F'\Delta F) \subseteq N$, then E' and F' satisfy the same relations as E and F.*

We denote by Σ^0 the set of null events, and observe the following properties of this set.

Claim 1. *The set of null events Σ^0 satisfies:*

1. $\emptyset \in \Sigma^0$;
2. *If N, M are in Σ^0 then also $N \cup M \in \Sigma^0$;*

[2] The symmetric difference of two events consists of all the states in these events that do not belong to both, that is, $A\Delta B = (A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$.

3. If $N \in \Sigma^0$ and event M satisfies $M \subseteq N$, then $M \in \Sigma^0$.

Proof. Part 1 and 3 follow trivially from the definition of a null event.

For part 2, assume N and M are null events, $E \succsim F$, and $(E'\Delta E) \cup (F'\Delta F) \subseteq N \cup M$. We need to show that $E' \succsim F'$.

By our assumptions there are events $N_0 \subseteq N$, $N'_0 \subseteq N$ and $M_0 \subseteq M$, $M'_0 \subseteq M$ such that $E \setminus E' = N_0 \cup M_0$ and $E' \setminus E = N'_0 \cup M'_0$. Let $\hat{E} = (E \cap E') \cup M_0 \cup N'_0$. Then, $(E \setminus \hat{E}) \subseteq N_0$, and $(\hat{E} \setminus E) \subseteq N'_0$, and hence $E\Delta\hat{E} \subseteq N$. We analogously define $\hat{F}$ such that $F\Delta\hat{F} \subseteq N$, and since N is null, we conclude that $\hat{E} \succsim \hat{F}$. Now, $E' \setminus \hat{E} \subseteq M'_0$ and $\hat{E} \setminus E' \subseteq M_0$, thus $\hat{E}\Delta E' \subseteq M$. A similar relation holds for $\hat{F}$ and F'. Thus, as M is null, we conclude that $E' \succsim F'$. The proof for the case that $E \not\succsim F$ is similar. □

The three properties of Σ^0 in Claim 1 make Σ^0 an *ideal* of events in Σ. Savage (1954) also proves that the null events defined in his model form an ideal. Finally, the set of null events of a probability measure is obviously an ideal.

Without making any assumption about $\succsim$, it is possible that all events are null. However, we next show that if this relation is not trivial, then there must be some non-null events.

Claim 2. *If there are vents E and F such that $E \succ F$, then there are non-null events.*

Proof. Assume that $E \succ F$ and suppose that contrary to the claim, all events are null. Set $E' = F$ and $F' = E$. Then $E'\Delta E$ and $F'\Delta F$ are null, and thus $E' \succsim F'$, that is, $F \succsim E$. Thus, $E \sim F$, contrary to our assumption that $E \succ F$. □

2.2 The axioms of desirability

Our first axiom requires that the desirability relation is non-degenerate. It is a mild assumption since without it there is nothing of interest to say about the given relation. Non-degeneracy assumption is assumed also in Savage (1954) as well as in the axioms of qualitative probability in de Finetti (1931).

Int 1. (Non-degeneracy) *There are events E and F such that $E \succ F$.*

In our setup, axiom Int 1 of Non-degeneracy guarantees, by Claim 2, that there are non-null events.

In the next axiom we require that the relation is a weak order of non-null events only. We denote the set of non-null events, $\Sigma \setminus \Sigma^0$, by Σ^+.

Int 2. (Weak Order) $\succsim$ *is contained in* $(\Sigma^+)^2$ *and it is a complete and transitive relation on the set of non-null events* Σ^+.

The completeness of a binary relation is a non-trivial assumption made also in Savage (1954) and de Finetti (1931). Here, we have the extra requirement that the order is defined only on the non-null events.

Next, we require that given a strict desirability relation between two events, the state space can be partitioned into events that are small in the sense that they do not effect the given relation. This axiom is a slight variation of property P6' in Savage (1954).

Int 3. (Non-atomicity) *For a given relation* $E \succ F$ *there exists a partition of* Ω, $\Pi = (\Pi_1, \ldots, \Pi_m)$, *such that for each* i, *if* $F' \Delta F \subseteq \Pi_i$, *then* $E \succ F'$, *and if* $E' \Delta E \subseteq \Pi_i$, *then* $E' \succ F$.

For $c \in \mathcal{C}$ we denote by C the event that the consequence of f is c. Namely, $C = \{\omega \mid f(\omega) = c\}$. We call the events C, *consequence events.* For each E and c we write E_c for $E \cap C$.

Int 4. (Pairs) *Let* E *and* F *be non-null events. If for each pair of distinct consequences,* c *and* d, $E_c \cup E_d$ *and* $F_c \cup F_d$ *are either both null or both non-null and* $E_c \cup E_d \sim F_c \cup F_d$, *then* $E \sim F$.

The Axiom of Pairs has no equivalent property for preferences on acts. However, a similar property holds for qualitative probability with respect to any partition of the state space.

The next three axioms capture the idea that the relation $\succsim$ describes desirability of events. They are not satisfied by qualitative probability and they do not correspond to properties of preference on acts in Savage's model. We illustrate these axioms with a simple example.

Consider a decision maker who is going to submit her paper to one of a few equally prestigious journals. A choice of a journal is an act. There are only three outcomes she cares about: the paper is accepted for publication (*acceptance*), she is required to revise and resubmit (*revision*), and the paper is rejected (*rejection*). Each of the acts results in each state of the world in one of the consequences. Suppose now that she chose to submit her paper to journal J. Her choice results in a unique consequence for each state.

The first axiom that is specific to desirability addresses the nature of consequence events that distinguishes them from other events. Such distinction does not exist in Savage's setup, as consequence events do not exist. The axiom says that when the agent is informed that a consequence c occurs then any additional information is irrelevant to desirability. In our example, compare the event *acceptance* with the event *acceptance and Alice is a referee*. The two events are equally desirable since once the paper is accepted the extra information that Alice is a referee does not matter to her. Hence the following axiom:

Int 5. (Consequence Events) *For any consequence c and a non-null event $E \subseteq C$, $E \sim C$.*

The following axiom formalizes the idea that a mixture of good news and bad news is more desirable than the bad news and less desirable than the good news. It has the same spirit as the averaging condition in Bolker (1967). We illustrate it with our previous example. Let E be the event *Alice was the the only referee* and F, which is disjoint from event E, that *Bob was the only referee*. Suppose that E is weakly more desirable than F, that is $E \succsim F$. The event $E \cup F$ is mixed news. Therefore E, the good news, must be at least as desirable as $E \cup F$, and $E \cup F$ must be at least as desirable as the less desirable event F.

Int 6. (Intermediacy) *Let E and F be disjoint non-null events. Then the relations $E \succsim F$, $E \cup F \succsim F$, and $E \succsim E \cup F$ are equivalent.*

As we have said, desirability of events is determined by the way they are related to the likelihood of consequences. Consider the events $A =$ *acceptance* and $B =$ *acceptance and Alice was a referee*, which by axiom Int 5 of Consequence Events are as desirable. Note, however, that as $B \subseteq A$, B is less likely than A. Now consider the event $G =$ *revision*, which is disjoint from A and B, and the events $A \cup G$ and $B \cup G$. The likelihood of *acceptance* in $A \cup G$ is higher than in $B \cup G$. Therefore $A \cup G$ should be more desirable than $B \cup G$. Note, that if G is an event disjoint from A and B that is less desirable than both, then we expect that the relation of desirability between $A \cup G$ and $B \cup G$ would be reversed.

The implication in this example is based on an informal, intuitive notion of likelihood relation. But we can reverse the reasoning and use such an implication to formally define a restricted concept of likelihood relation.

Definition 2. *Suppose that $A \sim B$, and G is a non-null event disjoint of A and B such that $G \nsim A$. Then A* is more likely than B given G *if either $E \succ G$ and $A \cup G \succsim B \cup G$, or $G \succ E$ and $B \cup G \succsim A \cup G$.*

If the relation of being more likely given G is to capture the likelihood of similar events then this relation should not depend on G. That is, if we take instead of G another event H with the same properties, then the relation of likelihood defined by H should be the same. This is the content of the next axiom which is in the spirit of the impartiality property in Bolker (1967).

Int 7. (Persistency) *Suppose* $A \sim B$*, and* G, H *are non null events disjoint of* A *and* B *such that* $G \not\sim A$ *and* $H \not\sim A$*. If* A *is more likely than* B *given* G*, then also* A *is more likely than* B *given* H*.*

2.3 *Ex ante* consistency of desirability relations

We will prove that a desirability relation for a given act f is represented by a probability-utility pair. However this representation is not unique. We consider next desirability relations for a family of acts and formulate five axioms concerning this family that guarantee the existence of a unique probability-utility pair that represents all the desirability relations in this family.

In the *ex ante* stage the agent considers various acts in a non-empty set $\mathcal{F}$. For each $f \in \mathcal{F}$ let $\succsim^f$ be a desirability relation on (Ω, Σ, f). We denote $\mathcal{D} = \{\succsim^f \mid f \in \mathcal{F}\}$. From now on we tag desirability relations in $\mathcal{D}$, as well as consequence events with a superscript of the act for which they are defined.

Ea 1. (Axioms of Desirability) *All the relations in* $\mathcal{D}$ *satisfy axioms* Int 1–Int 7*.*

Ea 2. (Common Null Events) *All the desirability relations in* $\mathcal{D}$ *have the same set of null events.*

By axiom Ea 2 of Common Null Events we can refer to null or non-null events without specifying a desirability relation in $\mathcal{D}$.

We say that an act f is *full*, if for each c, $C^f = f^{-1}(c)$ is non-null.

Ea 3. (Full Acts) *An act* f *is in* $\mathcal{F}$ *if and only it is full.*

The following axiom requires that the desirability relation between two events is independent of the value of the acts outside these events.

Ea 4. (Common Desirability) *Let* A *and* B *be non-null such that* $f|_A = g|_A$ *and* $f|_B = g|_B$*. Then,* $A \succsim^f B$ *if and only if* $A \succsim^g B$*.*

The last axiom says that when E and F are similar both by f and by g, then E is more likely than F by f if and only if E is more likely than F by g.

Ea 5. (Common Likelihood) *Let* E *and* F *be non-null events such that* $E \sim^f F$ *and* $E \sim^g F$*. Then, there exists* H *disjoint from* $E \cup F$ *such that* $H \not\sim^f E$ *and* $E \succsim^f_H F$ *if and only if there exists* H' *disjoint from* $E \cup F$ *such that* $H' \not\sim^g E$ *and* $E \succsim^g_{H'} F$*.*

3 The main theorems

Our first result concerns the representation of a desirability relation in a comprehensive state space. For this we define how a probability-utility pair represents a desirability relation.

Definition 3. (Representation) *Consider a pair* (P, u)*, where* P *is a non-atomic probability on* (Ω, Σ, f) *and* $u\colon \mathcal{C} \to \mathbb{R}$*. We say that* (P, u) represents *a binary relation* $\succsim$ *on* Σ *if the set of null events of* $\succsim$ *is the set of* P*-null events, and for all non-null events* A *and* B*,* $A \succsim B$ *if and only if*

$$\frac{\sum_{i=1}^{n} u(c_i)P(C_i \cap A)}{P(A)} \geq \frac{\sum_{i=1}^{n} u(c_i)P(C_i \cap B)}{P(B)}. \tag{1}$$

Obviously, both sides of inequality (1) are the conditional expectations of utility given the events A and B. The inequality can be written as $\sum_{i=1}^{n} u(c_i)P(C_i \mid A) \geq \sum_{i=1}^{n} u(c_i)P(C_i \mid B)$. Note, that Inequality (1) holds if we replace u by any positive affine transformation of u. That is $u \mapsto \alpha u + \beta$ where $\alpha > 0$.

Theorem 1. *For a comprehensive state space* (Ω, Σ, f)*, a relation* $\succsim$ *on* Σ *satisfies axioms* Int 1–Int 7 *if and only if there exists a pair* (P, u) *that represents it.*

In order to simplify the proof of Theorem 1, we first prove a special case of it where we make the following two assumptions.

Assumptions.

1. *for each consequence* c*, the event* C *is non-null,*
2. $C_n \succ C_{n-1} \succ \cdots \succ C_1$.

The main thrust of the second assumption is that no two distinct events C_i and C_j are similar. The ordering of desirability according to the indices is made, of course, without loss of generality.

Theorem 1 is now stated for the case that the two assumptions hold.

Theorem 1*. *For a comprehensive state space* (Ω, Σ, f)*, a relation* $\succsim$ *on* Σ *satisfies axioms* Int 1–Int 7 *and Assumptions* 1 *and* 2 *if and only if there exists a pair* (P, u) *that represents it, such that for* $i = 1, \ldots, n$*,* $P(C_i) > 0$ *and* $u(c_n) > u(c_{n-1}) > \cdots > u(c_1)$*.*

We illustrate the relation between probability-utility pairs and desirability relations in the following example.

Example 1. Let the state space (Ω, Σ) be the unit interval with the σ-algebra of Borel sets. The set of consequences is $\mathcal{C} = \{c_1, c_2, c_3\}$. The act f is defined by $f(\omega) = c_1$ for $\omega \in [0, 1/3)$, $f(\omega) = c_2$ for $\omega \in [1/3, 2/3)$, and $f(\omega) = c_3$ for $\omega \in [2/3, 1]$. Thus, the consequence events are: $C_1 = [0, 1/3)$, $C_2 = [1/3, 2/3)$, and $C_3 = [2/3, 1]$. The comprehensive state space is (Ω, Σ, f).

Consider the pair (P, u), where P is the uniform probability distribution, and the utility function, $u \colon \mathcal{C} \to \mathbb{R}$, is given by $u(c_1) = u_1 = 0$, $u(c_2) = u_2 = 1/2$, and $u(c_3) = u_3 = 1$. Denote by P_i, for $i = 1 \ldots 3$, the conditional probability of P on C_i. For a P-non-null event E, let $x_i = P(E|C_i)$. Then the conditional utility given E is:

$$[(0)(1/3)x_1 + (1/2)(1/3)x_2 + (1)(1/3)x_3]/[(1/3)x_1 + (1/3)x_2 + (1/3)x_3].$$

The conditional expectation defines a desirability relation $\succsim$ on the P-non-null events, which it represents. The null events of $\succsim$ are the P-null-events. Note, that the conditional expectation given E is determined by the x's. Thus, in particular, if two events have the same conditional probability given each C_i, then they are similar. Observe that for each i, $P(C_i) = 1/3 > 0$, and $u_1 < u_2 < u_3$ and thus, by Theorem 1*, Assumptions 1 and 2 hold.

The question that usually arises in representation theorems is the uniqueness of presentation. In our case the set of pairs that represent $\succsim$ is not a singleton. In the following theorems we characterize this set. We denote by $\mathcal{P}(\succsim)$ the set of all probability measures P such that for some u, (P, u) represents $\succsim$.

We decompose a probability P on (Ω, Σ) into two parts: The *conditional* part $(P_i)_{i=1}^n$ where for each i, $P_i(\cdot) = P(\cdot \mid C_i)$, and the *consequential* part, p, in the simplex $\Delta(\mathcal{C})$ where $p_i = P(C_i)$. Thus, for each event E, $P(E) = \sum_{i=1}^n p_i P_i(E)$. It turns out that the conditional part is uniquely determined for the given desirability relation, while the consequential part is not. In order to describe this non-uniqueness we introduce the notion of *optimism.*

For two positive probabilities p and q in $\Delta(\mathcal{C})$, we say that p is *more optimistic* than q, and write $q \gg q$ if for each $i < j$, $p_j/p_i > q_j/q_i$. The reason why this inequalities describe optimism follows from Assumption 2. If $p \gg q$, then for each two consequences the likelihood of the preferred one is higher in p than in q. Let $\rho(p)$ be the $n-1$ dimensional vector defined by $\rho_i(p) = p_{i+1}/p_i$ for $i = 1, \ldots n-1$. We say that p *likelihood-ratio* *dominates*[3] q if $\rho(p) > \rho(q)$. Obviously, p is more optimistic than q if and only if p likelihood-ratio dominates q.

An open interval of positive probabilities $(p, q) = \{\alpha p + (1-\alpha)q \mid 0 < \alpha < 1\}$ is *ordered by optimism* if for each $\alpha > \alpha'$, $\alpha p + (1-\alpha)q \gg \alpha' p + (1-\alpha')q$. The interval is *maximal* if p and q are on the boundary of the simplex.

We are now ready to describe the multiplicity of the probabilities in the representing pairs.

Theorem 2. *A set of probabilities $\mathcal{P}$ is $\mathcal{P}(\succsim)$ for some relation $\succsim$ on Σ that satisfies axioms* Int 1–Int 7 *and Assumptions* 1,2 *if and only if:*

1. *The conditional parts of the probabilities in $\mathcal{P}$ are the same. That is, for each P and Q in $\mathcal{P}$, $(P_i) = (Q_i)$,*

2. *The consequential parts of probabilities in $\mathcal{P}$ form a maximal interval ordered by optimism.*

Finally, we characterize the utilities in the representing pairs.

Theorem 3. *For every $P \in \mathcal{P}(\succsim)$, a utility u such that (P, u) represents $\succsim$ is uniquely determined up to a positive affine transformation of u.*

We can say more about the representing utilities. Denote $u_i = u(c_i)$ and define the vector of *utility gains* $\Delta u = (\Delta u_i)_{i=1}^{n-1}$ by $\Delta u_i = u_{i+1} - u_i$. By Theorem 1*, $\Delta u > 0$. For two utility vectors u and v we say that u is *more content* than v if for each $i < j$ between 2 and n, $\Delta u_j / \Delta u_i < \Delta v_j / \Delta v_i$. The $n-2$ dimensional vector $\rho(\Delta u)$, where $\rho_i(\Delta u) = \Delta u_{i+1} / \Delta u_i$ for $i = 2, \ldots, n-1$ is the vector of the *utility-gain ratio*. Obviously, u is more content than v if and only if $\rho(\Delta v) > \rho(\Delta u)$, that is, $\Delta v \gg \Delta u$. Note that $\rho(u)$ is invariant under positive affine transformations of u.

Roughly speaking, being more optimistic means giving higher probability to more desirable consequences, and being more content means giving less utility to such consequences. The next theorem says that being more optimistic is balanced by being more content.

Theorem 4. *For each $i = 2, \ldots, n-1$, the product $\rho_i(\Delta u)\rho_i(p)\rho_{i-1}(p)$ is the same for all (P, u) that represent $\succsim$. Thus, if (P, u) and (Q, v) represent $\succsim$, and $\rho(p) > \rho(q)$, then $\rho(\Delta u) < \rho(\Delta v)$.*

[3] It is straightforward to see that Likelihood-ratio dominance implies stochastic dominance.

Example 2. The desirability relation in Example 1 can be represented by other pairs (Q, v). By Theorem 2, the conditional probability of Q given each C_i is P_i. Thus, $Q = q_1 P_1 + q_2 P_2 + q_3 P_3$ for some probability vector $q = (q_1, q_2, q_3)$. If we choose $q = (1/6, 1/3, 1/2)$ and $v_1 = 0, v_2 = 3/4$ and $v_3 = 1$, then (Q, v) also represents $\succsim$. This can be easily verified by checking that the conditional expected utilities of the two pairs are similarly ordered. Note that $\rho(q) = (2, 3/2)$ while for $p = (1/3, 1/3, 1/3)$, in Example 1, $\rho(p) = (1, 1)$. Thus, $\rho(q) > \rho(p)$, and therefore q is more optimistic than p. Also $\Delta u = (1/2, 1/2)$, $\Delta v = (3/4, 1/4)$, and hence $\rho(\Delta u) = (1)$ and $\rho(\Delta v) = (1/3)$ which demonstrates Theorem 4.

In Section 5, we show how to compute the maximal interval of probability vectors that are ordered by optimism guaranteed by Theorem 2.

So far we have dealt with the representation of desirability for a given comprehensive state space, that is for a given act on a Savage state space. We referred to comprehensive state spaces as interim state spaces. We next consider an *ex ante* situation where the decision maker deliberates the choice of an act from a family of acts, and has a desirability relation for each of these acts. When the family of acts and the associated desirability relations satisfy axioms Ea 1–Ea 5, then there exists a single probability-utility pair that represents all the desirability relation in this family.

Theorem 5. *A family of acts $\mathcal{F}$ and a family of desirability relations $\mathcal{D} = \{\succsim^f \mid f \in \mathcal{F}\}$, satisfy axioms* Ea 1–Ea 5 *if and only if there exists a pair (P, u) that represents $\succsim^f$ for all acts $f \in \mathcal{F}$. Moreover, the probability P is uniquely determined and the utility vector u is uniquely determined up to a positive affine transformation.*

Example 3. Consider the state space of Example 1, and the pair (P, u) in this example. This pair defines a preference relation over all acts by taking the expected utility for each act. Moreover, by Savage (1954) this pair is the unique pair that represents this preference relation. Theorem 5 shows that this pair can be uniquely determined by desirability relations rather than a preference relation on acts.

Let $\mathcal{F}$ be the set of all acts f such that for each i, $f^{-1}(c_i)$ is P-non-null. For each $f \in \mathcal{F}$, the pair (P, u) defines a desirability relation $\succsim^f$ on events in the comprehensive state space (Ω, Σ, f). The family of these relations satisfies axioms Ea 1–Ea 5. By Theorem 5, the pair (P, u) is the only pair that satisfies these axioms.

4 Proofs

4.1 An outline of the proofs

We omit the proof of the simple "if" parts of Theorems 1 and 1*, and prove first the "only if" part of Theorem 1*.

In subsection 4.2 we derive for each consequence c a probability P_c on Σ_c, the σ-field of events in C, that will serve as the conditional probability of the probability P in Theorem 1*. Definition 2 enables us to define a likelihood relation on a family of similar events. Since, by axiom Int 5 of Consequence Events, all non-null subevents of C are similar, we manage to define a likelihood relation on Σ_c. This relation is shown to be a qualitative probability. By axiom Int 3 of Non-atomicity it follows by a theorem of Savage that there exists a unique non-atomic probability on Σ_c, which represents the qualitative likelihood relation on Σ_c.

In subsection 4.3 we show that the desirability relation between events depends only on the n-dimensional vector of their conditional probabilities $(P_c(E \cap C))_{c \in \mathcal{C}}$. Moreover, it is homogeneous in this vector.

This enables us to translate, in subsection 4.4, the desirability relation on events to a relation on the positive orthant of $\mathbb{R}^{\mathcal{C}}$. We show that the sets defined by this latter relation are convex, and characterize their topological properties.

In subsection 4.5 we again use Definition 2 to define a relation of being more likely on each equivalence class of points in $\mathbb{R}^{\mathcal{C}}$. We characterize the convexity of sets defined in terms of this relation and their topological properties.

We show in subsection 4.6 that the sets of being more likely than x and less likely than x, in the set of points equivalent to x, can be separated by a probability vector. Moreover this vector is independent of x. Such a probability vector will be the probability of the consequence events.

In subsection 4.7 we characterize the space of separating functionals of the previous subsection in terms of exchange rates of coordinates in the Euclidean space. These exchange rates help us to derive the utility in the next subsection.

Using the conditional utility in subsection 4.2, the probabilities derived in subsection 4.5, and the utility derived in subsection 4.8, we go back to the desirability relation and prove Theorems 1-4. In the last subsection we prove Theorem 5.

4.2 The conditional probability over consequences

The following are three immediate corollaries of axioms Int 6 of Intermediacy and Int 2 of Weak Order. The first is not only a corollary of the two axioms, but combined with axiom Int 2 implies axiom Int 6.

Corollary 2. *If E and F are disjoint non-null events, then the relations $E \succ F$, $E \cup F \succ F$, and $E \succ E \cup F$ are equivalent.*

Corollary 3. *If E and F are disjoint non-null events, then the relations $E \sim F$ and $E \cup F \sim F$ are equivalent. Hence, if $E^1, \ldots, E^k$ are non-null events that are disjoint in pairs, and $E^1 \sim E^2 \sim \cdots \sim E^k$, then $\cup_{i=1}^{k} E^i \sim E_1$.*

Corollary 4. *Let E and F be disjoint events. If $A \succ E$ and $A \succsim F$, then $A \succ E \cup F$. If $E \succ A$ and $F \succsim A$, then $E \cup F \succ A$.*

Proof. For the first part, if $E \succsim F$, then by intermediacy $A \succ E \succsim E \cup F$. If $F \succ E$, then by Corollary 1, $A \succsim F \succ E \cup F$. The second part is similarly proved. □

We denote by Σ_c the σ-algebra that Σ induces on C, namely, $\Sigma_c = \{E \mid E \subseteq C, E \in \Sigma\}$.

We begin with a derivation of a non-atomic probability distribution P_c on Σ_c for each consequence c. This is done by defining a relation $\gtrapprox$ on Σ_c, in terms of the relation $\succsim$, and showing that it satisfies the axioms of qualitative probability.

Fix for now a consequence c and the corresponding event C. Choose a non-null event G such that $G \cap C = \varnothing$ and $G \not\sim C$. By Assumption 2, and since $n \geq 2$, there exists such a G, as C_j for $j \neq i$ satisfies it. Note, that since G is non-null, for any $A \in \Sigma_c$, including the null events, $A \cup G$ is non-null. We define a binary relation $\gtrapprox$ on Σ_c as follows.

Definition 4. *For $A, B \in \Sigma_c$, $A \gtrapprox B$ if either $C \succ G$ and $A \cup G \succsim B \cup G$, or $G \succ C$ and $B \cup G \succsim A \cup G$.*

Observe, that non-null events A and B in Σ_c are similar events by axiom Int 5 of Consequence Events, and therefore $A \gtrapprox B$ if and only if A is more likely than B given G, as in Definition 2. Thus, $\gtrapprox$ is an extension of the latter relation to all events in Σ_c. We write $A \approx B$ when $A \gtrapprox B$ and $B \gtrapprox A$, and $A > B$ when it is not the case that $B \gtrapprox A$.

Proposition 1. *There exists a unique probability measure P_c on Σ_c such that for any $A, B \in \Sigma_c$, $A \gtrapprox B$ if and only if $P_c(A) \geq P_c(B)$. The probability P_c is non-atomic.*

Proof. We first show that $\gtrapprox$ is a qualitative probability on Σ_c. That is, it satisfies the following properties for all A, A', and B in Σ_c such that $B \cap (A \cup A') = \varnothing$.

1. $\gtrapprox$ is transitive and complete;

2. $A \gtrapprox A'$ if and only if $A \cup B \gtrapprox A' \cup B$;

3. $A \gtrapprox \varnothing$, $C > \varnothing$.

Since $G \not\sim C$, either $G \succ C$ or $C \succ G$. We assume that $C \succ G$. The proof for the other case is analogous.

By Weak Order either $A \cup G \succsim B \cup G$, in which case $A \gtrapprox B$, or $B \cup G \succsim A \cup G$, in which case $B \gtrapprox A$. Thus, $\gtrapprox$ is complete. Suppose that $A_1 \gtrapprox A_2$ and $A_2 \gtrapprox A_3$. Then, $A_1 \cup G \succsim A_2 \cup G \succsim$ and $A_2 \cup G \succsim A_3 \cup G$. By Weak Order $A_1 \cup G \succsim A_3 \cup G$, and thus $A_1 \gtrapprox A_3$. Therefore $\gtrapprox$ is transitive.

To show 2, we consider the following four cases. (a) B is null. In this case, $A \cup B \cup G \succsim A' \cup B \cup G$ if and only if $A \cup G \succsim A' \cup G$, which yields 2. (b) A is null and A' is not. This case is impossible when $A \gtrapprox A'$, because by Corollary 2, $A' \cup G \succ G \sim A \cup G$. (c) A is non-null and A' is null. By Intermediacy $A \cup G \succsim G \sim A' \cup G$. Thus, in this case, necessarily $A \gtrapprox A'$. Since $B \succ G$, $A \sim B \succsim B \cup G$, and hence by axiom Int 6 of Intermediacy, $A \cup B \cup G \succsim B \cup G \sim A' \cup B \cup G$. Thus, in this case it is also necessary that $A \cup B \gtrapprox A' \cup B$. (d) All three events A, A' and B are non-null. In this case, $A \gtrapprox A'$ means that A is more likely than A' given G. As $B \sim C \succ G$, it follows by Corollary 2 that $C \sim B \succ B \cup G$. Also $(B \cup G) \cap (A \cup A') = \varnothing$. Thus, by axiom Int 7 of Persistency, A is more likely than A' given G if and only if A is more likely than A' given $B \cup G$. Hence, $A \cup G \succsim A' \cup G$ iff and only if $A \cup B \cup G \succsim A' \cup B \cup G$.

If A is non-null, then by Corollary 2, $A \cup G \succ G = \varnothing \cup G$. Hence it is not the case that $\varnothing \cup G \succsim A \cup G$, and therefore $A > \varnothing$. In particular, $C > \varnothing$. If A is null then $A \cup G \sim \varnothing \cup G$. Which show that for all A, $A \gtrapprox \varnothing$. This proves 3.

Next, we prove a property of $\gtrapprox$ which is named by Savage P6':

If $E > F$, then there exists a finite partition of C, $(\Pi_i)_{i=1}^{k}$, such that for each i, $E > F \cup \Pi_i$.

Since $E > F$, it follows that $E \cup G \succ E \cup G$. Let $\{\Pi'_i \mid i = 1, \ldots, m\}$ be the partition the existence of which is guaranteed by axiom Int 3 of Non-atomicity for the last relation. Then, the set of nonempty events of the form $\Pi_i = \Pi'_i \cap C$ is a partition of C and for each such event Π_i, $(F \cup G \cup \Pi_i) \Delta (F \cup$

$G) = P_i \subseteq \Pi'_i$. Thus, by the said axiom, $E \cup G \succsim F \cup G \cup \Pi_i$, which means $E > F \cup \Pi_i$.

This property with the properties of $\gtrapprox$ as qualitative probability imply the claim of the proposition as is shown in Savage (1954). $\square$

In the next subsection we show that the desirability of an event E depends only on the probabilities $P_c(E_c)$. Here, we show that the question whether E is null or not depends only on these probabilities.

Definition 5. *Let* $\pi \colon \Sigma \to \mathbb{R}^{\mathcal{C}}$ *be defined by* $\pi(E) = (P_c(E_c))_{c \in \mathcal{C}}$.

Proposition 2. *An event* N *is null if and only* $\pi(N) = 0$.

Proof. Since Σ^0 is closed under unions, and inclusion, an event N is null if and only if for each c, N_c is null. Thus, it is enough to show that N_c is null if and only if $P_c(N_c) = 0$. If N_c is null then for any non-null H, $N_c \cup H \sim H$ and therefore $N_c \approx \varnothing$ and thus, $P_c(N_c) = 0$. For the converse suppose $P_c(N_c) = 0$. We need to show that if $E \succsim F$, $E \Delta E' \subseteq N_c$, and $F \Delta F' \subseteq N_c$ then $E' \succsim F'$. For this it suffices to show that $E \sim E'$ and $F \sim F'$. Note that $E \setminus C = E' \setminus C$. Now, if $E \setminus C \sim C$, then by Corollary 3 $E = E_c \cup (E \setminus C) \sim C$ and similarly $E' \sim C$ and we are done. Otherwise, $E \setminus C \not\sim C$. Now, $E_c = (E_c \cap E'_c) \cup N_c^1$ for some $N_c^1 \subseteq N_c$. Since $P_c(N_c^1) = 0$, it follows by axiom Int 7 of Persistency, that $E = (E_c \cap E'_c) \cup N_c^1 \cup (E \setminus C) \sim (E_c \cap E'_c) \cup (E \setminus C)$. Similarly $E' \sim (E_c \cap E'_c) \cup (E' \setminus C)$. Since $E \setminus C = E' \setminus C$, it follows that $E \sim E'$. Similarly, $F \sim F'$. $\square$

4.3 The homogeneity of desirability

In this subsection we prove:

Proposition 3. *If there exists* $t > 0$ *such that* $\pi(E) = t\pi(F) \neq 0$*, then* $E \sim F$.

To prove it we use the following three lemmas.

For each non-null G, the *support* of G is $\mathcal{C}(G) = \{c \mid G_c \text{ is non-null}\}$. We split the support into two parts $C^-(G) = \{c \in \mathcal{C}(G) \mid G \succ G_c\}$ and $\mathcal{C}^+(G) = \{c \in \mathcal{C}(G) \mid G_c \succsim G\}$.

Lemma 1. *The set* $\mathcal{C}^+(G)$ *is not empty, and if* $|\mathcal{C}(G)| \geq 2$*, then also* $\mathcal{C}^-(G)$ *is not empty.*

Proof. Suppose that $C^+(G) = \varnothing$. Then $G = \cup_{c \in C^-(G)} G_c$. By Corollary 4, $G \succ \cup_{c \in C^-(G)} G_c$, which is impossible. Assume now that $|\mathcal{C}(G)| \geq 2$ and suppose that $C^-(G) = \varnothing$. Then for some c and d in $C^+(G)$, $G_c \succ G_d$. Again by Corollary 4, $G = \cup_{c \in C^+(G)} G_c \succ G$. $\square$

Lemma 2. *Let G be an event such that $|\mathcal{C}(G)| \geq 2$. Denote for each event X such that $\mathcal{C}(X) = \mathcal{C}(G)$, $X^+ = \cup_{c \in \mathcal{C}^+(G)} X_c$ and $X^- = \cup_{c \in \mathcal{C}^-(G)} X_c$. If $G^+ \subset X^+$ and $X^- \subset G^-$, and the events $G^+ \backslash X^+$ and $X^- \backslash G^-$ are non-null, then $X \succ G$.*

Proof. By Corollary 4, $G \succ G^- \setminus X^-$. This implies that $X^- \cup G^+ \succ G$, because if $G \succsim X^- \cup G^+$, then $G \succ (X^- \cup G^+) \cup (G^- \setminus X^-) = G$. Also, $X^+ \setminus G^+ \succsim G$. Hence, $(X^- \cup G^+) \cup (X^+ \setminus G^+) \succ G$. Since $\mathcal{C}(X) = \mathcal{C}(G)$ it follows that $X \sim (X^- \cup G^+) \cup (X^+ \setminus G^+)$ and thus $X \succ G$. □

Next, we describe a simple result of axiom Int 3 of Non-atomicity. If $F \succ E$, and $E^1 \subseteq E$ is non-null, then there exists $D \subseteq E^1$ such that $D \cap E^1$ is non-null and $F \succ E \backslash D$. Indeed, choose the partition Π in axiom Int 3, and select an element Π_i of Π such that $\Pi_i \cap E^1$ is non-null, and set $D = \Pi_i \cap E^1$. This result can be generalized as follows.

Lemma 3. *If $F \succ E$, and $E^1, ..., E^m$ are non-null subevents of E. Then there exists $D \subseteq \cup_{i=1}^m E^i$ such that for each i, $D \cap E^i$ is non-null and $F \succ E \setminus D$.*

Proof. Prove by induction on m. In the k stage we have D^k that satisfies the condition for $E^1, ..., E^k$. Since $F \succ E \setminus D^k$, we can apply axiom Int 3 of Non-atomicity and choose P_i such that $P_i \cap E^{k+1}$ is non-null. We let $D^{k+1} = (D^k \cup P_i) \cap \cup_{i=1}^{k+1} E^i$. □

Proof of Proposition 3. By Proposition 2, $\mathcal{C}(E) = \mathcal{C}(F) = \{c \mid p_c(E_c) = p_c(F_c) > 0\}$. If this set, which we denote by $\mathcal{C}$, is a singleton c, then both E and F are similar to C and we are done. We assume therefore that $|\mathcal{C}| \geq 2$.

We prove first for $t = 1$. By the definition of p_c and axiom Int 7 of Persistency, for each $d \neq c$ in $\mathcal{C}$, $E_c \cup F_d \sim F_c \cup F_d$. Similarly, by the definition of p_d, $E_c \cup F_d \sim E_c \cup E_d$. Thus, $E_c \cup E_d \sim F_c \cup F_d$. It follows by axiom Int 4 of Pairs that $E \sim F$.

Suppose that $t = k/m$ for some integers k and m. By the non-atomicity of p_c, there exists for each $c \in \mathcal{C}$, a partition $E_c^1, \ldots, E_c^k$ of E_c into k equally p_c-probable events and a partition $F_c^1, \ldots, F_c^m$ of F_c into m equally p_c-probable events. Then $p_c(E_c^i) = p_c(F_c^j)$ for all $c \in \mathcal{C}$ and i, j. Let $E^i = \cup_{c \in \mathcal{C}} E_c^i$ and $F^j = \cup_{c \in \mathcal{C}} F_c^j$. Then, by the claim for $t = 1$, $E^i \sim F^j$ for all i and j. As all the E^i's are disjoint in pairs and similar, it follows by Corollary 3 that $\cup_{i=1}^k E^i \sim E^1$. In the same way, $\cup_{j=1}^m F^j \sim F^1$. Since for all $c \notin \mathcal{C}$, E_c and F_c are null, $E \sim \cup_{i=1}^k E^i$ and $F \sim \cup_{j=1}^m F^j$. But, $E^1 \sim F^1$, and therefore $E \sim F$.

Let t be an irrational number. Suppose that contrary to the claim, $F \succ E$. This can be assumed without loss of generality, because if $E \succ F$ we write $\pi(F) = t'\pi(E)$ for $t' = 1/t$.

We derive a contradiction. By Lemma 1, $\mathcal{C}^-(F)$ is not empty. By Lemma 3, there exists an event D such that $F \succ E \setminus D$, $D \subseteq \cup_{c \in \mathcal{C}^-(F)} E_c$, and $D \cap E_c$ is non-null for each $c \in \mathcal{C}^-(F)$. We denote $H_c = E_c \setminus D$. Let $\varepsilon = \min\{p_c(E_c \cap D) \mid c \in \mathcal{C}^-(F)\}$. Then, $\varepsilon > 0$ and we can choose a rational number k/n such that $t - \varepsilon < k/m < t$. Given this relation we have by the non-atomicity of the probabilities p_c an event $G \subseteq E$ such that $\pi(G) = (k/m)\pi(F)$. Moreover, for $c \in \mathcal{C}^-(F)$, we can choose G_c to satisfy $H_c \subseteq G_c$ where the difference is a non-null event.

As we have shown, $G \sim F$. Therefore, if $F \succ F_c$ then $G \sim F \succ F_c \sim G_c$. Thus, $\mathcal{C}^-(G) = \mathcal{C}^-(F)$, and similarly, $\mathcal{C}^+(G) = \mathcal{C}^+(F)$. We apply Lemma 2 to $X = E \setminus D$. The event X^- is $H = \cup_{c \in \mathcal{C}^-(G)} H_c \subset G^-$, and $X^+ = E^+ \supset G^+$. We conclude that $F \succ E \setminus D \succ G \sim F$ which is a contradiction. □

4.4 From desirability to a relation in a Euclidian space

Using Proposition 3, we describe a binary relation on $\mathbb{R}^{\mathcal{C}}$. We use the notation $\succsim$ for both this relation and the relation on events, and call both desirability relations. No confusion will result.

Definition 6. *Denote by $\mathbb{R}^{\mathcal{C}}_+$ the set of all point $x \in \mathbb{R}^{\mathcal{C}}$ such that $x \geq 0$ and $x \neq 0$. We define a relation on $\mathbb{R}^{\mathcal{C}}_+$ by $x \succsim y$ if there exist events E and F and positive numbers t and s such that $\pi(E) = tx$, $\pi(F) = sy$, and $E \succsim F$.*

Note that if $x \succsim y$ then by Proposition 3, $E' \succsim F'$ for any pair of events E' and F' such that $\pi(E') = t'x$ and $\pi(F') = s'y$, for $t', s' > 0$.

Denote $\mathcal{M}(x) = \{y \mid y \succsim x\}$, $\mathcal{M}_+(x) = \{y \mid y \succ x\}$, $\mathcal{L}(x) = \{y \mid x \succsim y\}$, $\mathcal{L}_-(x) = \{y \mid x \succ y\}$, and $\mathcal{E}(x) = \{y \mid y \sim x\}$.

The next proposition addresses the convexity of these sets.

Proposition 4.

1. *The relation $\succsim$ on $\mathbb{R}^{\mathcal{C}}_+$ is complete and transitive.*

2. *For each x, the sets $\mathcal{M}(x)$, $\mathcal{M}_+(x)$, $\mathcal{L}(x)$, $\mathcal{L}_-(x)$, and $\mathcal{E}(x)$ are convex cones.*

Proof. 1. For x and y in $\mathbb{R}^{\mathcal{C}}$ there exist small enough positive t and s such that for some events E and F, $\pi(E) = tx$ and $\pi(F) = sy$. Since at least one of the relations $E \succsim F$ or $F \succsim E$ holds, it follows that at least one of $x \succsim y$ or $y \succsim x$ must hold.

Suppose $x \succsim y$ and $y \succsim z$. Then there are events E, F, and positive numbers t_E and t_F, such that $\pi(E) = t_E x$, $\pi(F) = t_F y$, and $E \succsim F$. There are also events G and H, and positive numbers t_H and t_G, such that $\pi(G) = t_G y$, and $\pi(H) = t_H z$, where $G \succsim H$. Since $\pi(G) = t_G t_H^{-1} \pi(H)$, it follows by Propostion 3 that $G \sim H$. Hence, $E \succsim H$ and therefore $x \succsim z$.

2. The sets in this part of the proposition are cones by the definition of $\succsim$. Consider the set $\mathcal{M}(x)$. To prove that it is convex it is enough to show that for any $z, w \in \mathcal{M}(x)$, $z + w \in \mathcal{M}(x)$. Let G be an event such that $\pi(G) = rx$. For small enough $t > 0$ there are disjoint events E and F such that $\pi(E) = tz$ and $\pi(F) = tw$. Hence, $E \succsim G$ and $F \succsim G$. By Corollaries 3 and 4, $E \cup F \succsim G$. But $\pi(E \cup F) = t(z + w)$ and thus $z + w \in \mathcal{M}(x)$. The proof for the rest of the sets is similar. □

Next, we discuss the topological properties of these sets. We denote by e_c the unit vector of the coordinate c, and write e_i for e_{c_i}.

Proposition 5. *For each $x \in \mathbb{R}_+^{\mathcal{C}}$:*

1. *the sets $\mathcal{M}_+(x)$ and $\mathcal{L}_-(x)$, are open subsets in $\mathbb{R}_+^{\mathcal{C}}$. If $x \neq e_1$ then $\mathcal{L}_-(x) \neq \varnothing$. If $x \neq e_n$ then $\mathcal{M}_+(x) \neq \varnothing$;*
2. *the sets $\mathcal{M}(x)$, $\mathcal{L}(x)$, and $\mathcal{E}(x)$ are closed subsets in $\mathbb{R}_+^{\mathcal{C}}$;*
3. *the interior of $\mathcal{E}(x)$ is empty.*

Proof. 1. Let $y \in \mathcal{M}_+(x)$ and suppose that $\pi(E) = ty$ and $\pi(F) = sx$. We may assume without loss of generality that $ty_c < 1$ for each c. As $E \succ F$ we can apply axiom Int 3 of Non-atomicity. Consider a consequence c. If $P_c(E_c) > 0$, then E_c is non-null, and we can find an element Π_i of the partition Π such that $\Pi_i \cap E_c$ is non-null. Denote $D_c = E_c \cap \Pi_i$. Then $E \backslash D_c \succ F$. As $\pi(E \backslash D_c) = ty - p_c(D_c)e_c$, it follows that $y - t^{-1}p_c(D_c)e_c \succ x$. Thus, at a point y which is not on the face $y_c = 0$, we can decrease the c-coordinate and remain in $\mathcal{M}_+(x)$. Similarly, since $C \setminus E_c$ is non-null, per our assumption on ty, we can choose an element Π_i of the partition Π, such that $(C \setminus E_c) \cap \Pi_i$ is non-null. By setting $D_c = (C \setminus E_c) \cap \Pi_i$, we have $E \cup D_c \succ F$. In this way we show that $y + t^{-1}p_c(D_c)e_c \succ x$. Thus, we can increase the c-coordinate and remain in $\mathcal{M}_+(x)$. Since $\mathcal{M}_+(x)$ is convex, to prove that it is open it is enough to show that for each point y in $\mathcal{M}_+(x)$ an interval along the c-coordinate containing y is in $\mathcal{M}_+(x)$. If $x \neq e_n$, then $e_n \succ x$ and hence $\mathcal{M}_+(x)$ is not empty. The proof for the set $\mathcal{L}_-(x)$ is similar.

2. The sets $\mathcal{M}(x)$ and $\mathcal{L}(x)$ are the complements in $\mathbb{R}^{\mathcal{C}}_+$ of $\mathcal{L}_-(x)$ and $\mathcal{M}_+(x)$ correspondingly, and hence they are closed. The set $\mathcal{E}(x)$ is the intersection of $\mathcal{M}(x)$ and $\mathcal{L}(x)$ and hence closed.

3. Let $y \in \mathcal{E}(x)$. There exists c such that either $y \succ e_c$ or $e_c \succ y$. Suppose the first holds. We can assume without loss of generality that $y = \pi(E)$ and $y_c < 1$. Choose $F_c \subseteq C$, such that $F_c \cap E = \varnothing$ and $p_c(F_c) < \varepsilon$. Then $E \succ E \cup E_c$. This means that $y \succ y + \varepsilon e_c$, and therefore $y + \varepsilon e_c \notin \mathcal{E}(x)$. This shows that y is not in the interior of this set. The proof for the case $e_c \succ y$ is similar. □

For $x \notin \{e_1, e_n\}$, the three sets $\mathcal{M}_+(x)$, $\mathcal{L}_-(x)$ and $\mathcal{E}(x)$ form a partition of $\mathbb{R}^{\mathcal{C}}_+$. The first two are disjoint open convex cones. Since $\mathcal{E}(x)$ does not have an interior point, it is the closure of each of the first two sets. These two convex open sets can be separated by a hyperplane. Since 0 is in the closure of the separated sets, the hyperplane is an $(n-1)$-dimensional subspace $\mathcal{S}(x)$. As $\mathcal{E}(x)$ is the closure of both sets, it must be the intersection of $\mathcal{S}(x)$ with $\mathbb{R}^{\mathcal{C}}_+$. Since the two separated sets are open, $\mathcal{E}(x)$ contains an interior point of $\mathbb{R}^{\mathcal{C}}_+$. Thus we conclude:

Corollary 5. *For $x \notin \{e_1, e_n\}$, the set $\mathcal{E}(x)$ is the intersection of $\mathbb{R}^{\mathcal{C}}_+$ with an $(n-1)$-dimensional subspace, $\mathcal{S}(x)$. This intersection is of dimension $n-1$, that is, it contains interior points of $\mathbb{R}^{\mathcal{C}}_+$.*

4.5 Likelihood relation in the Euclidean space

Using the desirability relation of events we defined a likelihood relations $\succsim_H$ on events which are equally desirable. We now show how such relations are transformed to a relation in $\mathbb{R}^{\mathcal{C}}$.

For $v \not\sim x$ we define a relation $\succsim^*_v$ on $\mathcal{E}(x)$.

Definition 7. *For $y, z \in \mathcal{E}(x)$, if $x \succ v$, then $y \succsim^*_v z$ when $y + v \succsim z + v$, and if $v \succ x$ then $y \succsim^*_v z$ when $z + v \succsim y + v$.*

By axiom Int 7 of Persistency, if $u, v \not\sim x$ then $\succsim^*_u = \succsim^*_v$. We denote this relation which is independent of the choice of v, by $\succsim^*$. We study the following sets that are defined in terms of this relation.

For each $y \in \mathcal{E}(x)$, we define five subsets of $\mathcal{E}(x)$: $\mathcal{M}^*(y) = \{z \mid z \succsim^* y\}$, $\mathcal{M}^*_+(y) = \{y \mid z \succ^* y\}$, $\mathcal{L}^*(y) = \{z \mid y \succsim^* z\}$, $\mathcal{L}^*_-(y) = \{z \mid y \succ^* z\}$, and $\mathcal{E}^*(y) = \{z \mid z \sim^* y\}$.

First, we describe the convexity properties of these sets.

Proposition 6.

1. *The relation* $\succsim^*$ *on* $\mathcal{E}(x)$ *is complete and transitive.*

2. *For each* $y \in \mathcal{E}(x)$*, the sets* $\mathcal{M}^*(y)$*,* $\mathcal{M}^*_+(y)$*,* $\mathcal{L}^*(y)$*,* $\mathcal{L}^*_-(y)$*, and* $\mathcal{E}^*(y)$ *are convex.*

Proof. 1. Since either $y + v \succsim z + v$ or $z + v \succsim y + v$, it follows that either $y \succsim_v z$ or $z \succsim_v y$. Suppose $y \succsim_v z$ and $z \succsim_v w$. Then $y + v \succsim z + v \succsim w + v$ and therefore $y \succsim_v w$.

2. Let $z, w \in \mathcal{M}^*(y)$ and $\alpha \in (0, 1)$. Then for some v such that $x \succ v$, $z + v \succsim y + v$ and $w + v \succ y + v$. Therefore, $\alpha z + \alpha v \succsim y + v$, and $(1 - \alpha)w + (1 - \alpha)v \succsim y + v$. By intermediacy, $\alpha z + (1 - \alpha)w + v \succsim y + v$. That is, $\alpha z + (1 - \alpha)w \in \mathcal{M}^*(y)$. The proof for the rest of the sets is similar. □

The following lemma is used in the next proposition that describes the topological properties of these sets.

Lemma 4. *For all* $y, z \in \mathcal{E}(x)$:

1. $z + y \succ^* y$;

2. *if* $y \sim^* z$ *and* $t > 0$ *then* $ty \sim^* tz$.

Proof. 1. Let $x \succ v$. By intermediacy, $z \sim y \succ y + v$. Therefore, $z + y + v \succ y + v$. Hence $z + y \succ^* y$.

2. If $y \sim^* z$ then for some v such that $x \succ v$, $y + v \sim z + v$. Therefore, $ty + tv \sim tz + tv$ and thus $ty \sim^* tz$. □

Proposition 7. *For each* $y \in \mathcal{E}(x)$:

1. *the sets* $\mathcal{M}^*_+(y)$ *and* $\mathcal{L}^*_-(y)$*, are non-empty open subsets in* $\mathcal{E}(x)$;

2. *the sets* $\mathcal{M}^*(y)$*,* $\mathcal{L}^*(y)$*, and* $\mathcal{E}^*(y)$ *are closed subsets in* $\mathcal{E}(x)$;

3. *the interior of* $\mathcal{E}^*(y)$ *in* $\mathcal{E}(x)$ *is empty.*

Proof. 1. By Lemma 4, $y + \varepsilon y \succ^* y \succ^* y - \varepsilon y$ and thus $\mathcal{M}^*_+(y)$ and $\mathcal{L}^*_-(y)$ are not empty. This also shows that close enough to $\mathcal{E}^*(y)$ there are points not in this set, which proves 3. If $z + v \succ y + v$, then by Proposition 5 there is a ball B around $z + v$ such that for each $w \in B$, $w \succ y + v$. Therefore, there is a ball B' around y such that for each $w' \in B'$, $w' + v \succ y + v$. Thus $y \in B' \cap \mathcal{E}(x)$ which shows that $\mathcal{M}^*_+(y)$ is open. The proof for $\mathcal{L}^*_-(y)$ is similar.

2. The first two sets are complements of open sets, and the third is the intersection of the first two. □

4.6 Separation

By Propositions 6 and 7 we can separate $\mathcal{M}^*(y)$ and $\mathcal{L}^*(y)$ by a hyperplane. Since $\mathcal{E}^*(y)$ is the boundary of each of these sets it is contained in this hyperplane. As the separated sets are of dimension $n-1$, $\mathcal{E}^*(y)$ is of dimension $n-2$. Thus,

Corollary 6. *For $y \in \mathcal{E}(x)$, there exists a unique subspace $L(x,y)$ of dimension $n-2$ such that $\mathcal{E}^*(y) = (L(x,y)+y) \cap \mathcal{E}(x)$*

We next show in two steps that the space $L(x,y)$ is independent of x and y.

Proposition 8. *There exists an $(n-2)$-dimensional subspace L such that for all x and $y \in \mathcal{E}(x)$, $L(x,y) = L$.*

We prove it with the next three lemmas. We first fix x and vary y.

Lemma 5. *For each x there exists $L(x)$ such that for all $y \in \mathcal{E}(x)$, $L(x,y) = L(x)$.*

Proof. Let $y' \in \mathcal{E}(x)$. By the separation, the ray ty must intersect $\mathcal{E}^*(y')$, and thus, for some $t > 0$, $y' \sim^* ty$ and hence $\mathcal{E}^*(y') = \mathcal{E}^*(ty)$. By Lemma 4, $\mathcal{E}^*(ty) = t\mathcal{E}^*(y)$. But $t\mathcal{E}^*(y) = t[L(x,y)+y) \cap \mathcal{E}(x)] = (L(x,y)+ty) \cap \mathcal{E}(x)$. Thus, $L(x,y') = L(x,y)$. □

In order to show that $L(x)$ is independent of x we use the next lemma.

Lemma 6. *For each x,y, and z, if $x \sim^* y$, then $x+z \sim^* y+z$.*

Proof. If $x \sim^* y$, then by definition $x \sim y$. Suppose $x \succ z$. As $x \sim^* y$ it follows that $x+z \sim y+z$. In order to show that $x+z \sim^* y+z$ it is enough to find some v such that $x+z \succ v$, and $x+z+v \sim y+z+v$. Indeed, take $v = z$, then by Intermediacy $x+z \succ z$, and as $x \sim^* y$, $x+(z+z) \sim y+(z+z)$. The proofs for the cases that $z \succ v$ and $z \sim z$ are similar. □

Lemma 7. *There exists L such that for all x, $L(x) = L$.*

Proof. For x and x' choose $y \in \mathcal{E}(x)$ and $y' \in \mathcal{E}(x')$ such that $y'-y = z \in \mathbb{R}^{\mathcal{C}}_+$. By Lemma 6, $\mathcal{E}^*(y)+z \subseteq \mathcal{E}^*(y')$. But, $\mathcal{E}^*(y') = (L(x')+y') \cap \mathcal{E}(x')$, and $\mathcal{E}^*(y)+z$ is an $(n-2)$-dimensional subset of $L(x)+y+z = L(x)+y'$. Therefore, $L(x) = L(x')$. □

This completes the proof of Proposition 8.

Since L is of dimension $n-2$ there are many linear functionals p such that $pw = 0$ for all $w \in L$. By the definition of L, each such functional separates $\mathcal{M}^*(y)$ and $\mathcal{L}^*(y)$, and contains $\mathcal{E}^*(y)$ for every x and $y \in \mathcal{E}(x)$. The separating functional p is going to play the role of consequential probabilities. Therefore we need the following claim.

Proposition 9. *The functional p can be chosen to be a strictly positive probability vector.*

Proof. Let p' be a separating functional. By Lemma 4, for fixed x and $y \in \mathcal{E}(x)$, and for any $w \in \mathcal{E}(x)$, $y + w \in \mathcal{M}^*_+(y)$. Therefore, $p'(y+w) \neq p'y$ and thus $p'w \neq 0$.

Since $\mathcal{E}(x)$ is the intersection of $\mathbb{R}^{\mathcal{C}}_+$ with a subspace $\mathcal{S}$ of dimension $n-1$, there exists a non-zero functional $q \in \mathbb{R}^{\mathcal{C}}$ such that for each $w \in \mathbb{R}^{\mathcal{C}}_+$, $qw = 0$ if and only if $w \in \mathcal{E}(x)$.

Consider the two-dimensional space $\alpha p' + \beta q$. We show that it contains a point in $\mathbb{R}^{\mathcal{C}}_+$. Suppose to the contrary that $\{\alpha p' + \beta q \mid \alpha, \beta \in \mathbb{R}\} \cap \mathbb{R}^{\mathcal{C}}_+ = \emptyset$. Then, the two sets can be separated by a non-zero functional w. Since the first set is a subspace, $w(\alpha p' + \beta q) = 0$ for each α and β, and we can assume that $wr \geq 0$ for all $r \in \mathbb{R}^{\mathcal{C}}_+$ which implies that $w \in \mathbb{R}^{\mathcal{C}}_+$. By the separation, $wq = 0$ and $wp' = 0$. The first equality implies that $w \in \mathcal{E}(x)$. But then the second equation is impossible because we proved that $p'w \neq 0$ for each $w \in \mathcal{E}(x)$. Therefore, we can choose $p = \alpha p' + \beta q$ in $\mathbb{R}^{\mathcal{C}}_+$. By the definition of q, for every $z \in \mathcal{E}^*(y)$, $pz = \alpha p'z = \alpha p'y = py$, which shows that p vanishes on L.

To see that p is strictly positive, note that for e_c, $pe_c = p_c$. By Lemma 4, $e_c + e_c \succ^* e_c$ and therefore $2p_c > p_c$, which shows that $p_c > 0$. We can assume that p is normalized and therefore it is a strictly positive probability vector. □

4.7 The family of separating functionals

When $n = 2$ the dimension of L is 0. The probability vector p can be chosen in this case to be any vector $(a, 1-a)$ for $0 < a < 1$. We now assume that $n > 2$ and construct a basis for L.

Proposition 10. *For each $i = 2, \ldots, n-1$ there is a unique pair of positive numbers δ_i, η_i, such that the vector d^i, defined by $(d^i_{i-1}, d^i_i, d^i_{i+1}) = (\delta_i, -1, \eta_i)$ and $d_j = 0$ for all $j \notin \{i-1, i, i+1\}$, is in L. The vectors d^i form a basis of L.*

Proof. For $i = 2, \ldots, n-1$, let $\mathbb{R}(i)$ be the subspace of $\mathbb{R}^{\mathcal{C}}$ spanned by e_{i-1}, e_i, and e_{i+1}, and $\mathbb{R}_+(i) = \mathbb{R}(i) \cap R_+^{\mathbb{C}}$. Since the dimension of L is $n-2$, the dimension of $L \cap \mathbb{R}(i)$ is at least 1, and it cannot be higher than 1 because then there are $x, y \in \mathbb{R}(i)$ such that $x > y$ and $x - y \in L$, contrary to Lemma 4. Thus, $L \cap \mathbb{R}(i)$ is of dimension 1.

Choose two distinct points x and y in the interior of $\mathbb{R}_+(i)$ such $x - y \in L$. We show that $x_i \neq y_i$. Suppose to the contrary that $x_i = y_i$. Since $x - y \in L$, it follows that $p_{i-1}(y_{i-1} - x_{i-1}) + p_{i+1}(y_{i+1} - x_{i+1}) = 0$. Since $p > 0$, $y_{i-1} - x_{i-1}$ and $y_{i+1} - x_{i+1}$ are of different signs. But $e_{c_{i-1}} \succ x \succ e_{c_{i+1}}$ and thus by Lemma 2 either $y \succ x$ or $x \succ y$, which contradicts the assumption that $x \sim y$.

Thus, we can assume without loss of generality that $y_i < x_i$. Now, $p_{i-1}(y_{i-1} - x_{i-1}) + p_i(y_i - x_i) + p_{i+1}(y_{i+1} - x_{i+1}) = 0$, and since the middle term is negative, $p_{i-1}(y_{i-1} - x_{i-1}) + p_{i+1}(y_{i+1} - x_{i+1}) > 0$. Thus it is impossible that $y_{i-1} - x_{i-1} \leq 0$ and $y_{i+1} - x_{i+1} \leq 0$. Also, as $e_{c_{i-1}} \succ y \succ e_{c_{i+1}}$, it is impossible that one difference is positive and the other is non-negative, because this would imply contrary to $x \sim y$, that either $y \succ x$ or $x \succ y$. Therefore both are positive. Let,

$$\delta_i = \frac{y_{i-1} - x_{i-1}}{x_i - y_i} \tag{2}$$

and

$$\eta_i = \frac{y_{i+1} - x_{i+1}}{x_i - y_i}. \tag{3}$$

Then $y - x = (x_i - y_i)d^i$. Since $x - y \in L$, it follows that $d^i \in L$. Since $L \cap \mathbb{R}(i)$ is a line, δ_i and η_i are uniquely determined.

Since the vectors $d^2, \ldots, d^{n-1}$ are $n-2$ independent vectors they are a basis of L. □

Corollary 7. *For $i = 2, \ldots, n-1$,*

$$\delta_i p_{i-1} + \eta_i p_{i+1} = p_i. \tag{4}$$

4.8 Utility

We now construct a utility vector $u = (u_c)$, where we write u_i for u_{c_i}. We say that u is *monotonic* if $u_i < u_{i+1}$ for $i = 1, \ldots, n-1$.

Proposition 11. *There exists a monotonic vector u such the function*

$$\hat{u}(x) = \sum_c p_c x_c u_c / px$$

on $\mathbb{R}^{\mathcal{C}}_+$ is constant on $\mathcal{E}(x^0)$, for each $x^0 \in \mathbb{R}^{\mathcal{C}}_+$. The vector u is uniquely determined up to transformations $u \to \alpha(u_1+\beta, u_2+\beta, \ldots, u_n+\beta)$, for $\alpha > 0$.

Proof. When $n = 2$, $\mathcal{E}(x^0)$ is simply the ray $\{tx^0 \mid t > 0\}$. Since $\hat{u}$ is homogeneous, the claim of the proposition holds for any monotonic vector (u_1, u_2). Assume now that $n > 2$.

Consider first $x \in \mathcal{E}^*(x^0)$. Since $px = px^0$, $\hat{u}(x) = \hat{u}(x^0)$ is equivalent to

$$\sum_c p_c(x_c - x^0_c)u_c = 0.$$

By Proposition 10, for small enough t, $x = x^0 + td^i \in \mathcal{E}^*(x^0)$. The last equality in this case is equivalent to:

$$\delta_i p_{i-1}u_{i-1} + \eta_i p_{i+1}u_{i+1} = p_i u_i. \tag{5}$$

Using equation (4), equation (5) can be written as

$$\delta_i p_{i-1}(u_{i-1} - u_{i-1}) + \eta_i p_{i+1}(u_{i+1} - u_{i-1}) = p_i(u_i - u_{i-1}). \tag{6}$$

This gives rise to: $(u_{i+1} - u_{i-1})/(u_i - u_{i-1}) = p_i/(\eta_i p_{i+1})$. Denoting $\Delta u_i = u_i - u_{i-1}$ for $i = 2, \ldots, n$, Equation (6) is $(\Delta u_{i+1} + \Delta u_i)/\Delta u_i = p_i/(\eta_i p_{i+1})$, or

$$\frac{\Delta u_{i+1}}{\Delta u_i} = \frac{p_i}{\eta_i p_{i+1}} - 1 = \frac{\delta_i p_{i-1}}{\eta_i p_{i+1}}, \tag{7}$$

where the right-hand side is positive. Thus, choosing arbitrarily $u_1 < u_2$, the rest of the coordinates of u are determined by 7, and as the Δu_i's are positive, u is monotonic. Obviously, a vector v solves (5) if and only if for some $\beta \in \mathbb{R}^{\mathcal{C}}$ and a positive α, $v = \alpha(u_1 + \beta, u_2 + \beta, \ldots, u_n + \beta)$.

Now, considering tx^0. Obviously, $\hat{u}(tx^0) = \hat{u}(x^0)$. Thus the function $\hat{u}$ is constant on $\cup_{t>0}\mathcal{E}^*(tx^0)$, which is $\mathcal{E}(x^0)$. □

Proposition 12. *$x \succsim y$ if and only if $\hat{u}(x) \geq \hat{u}(y)$.*

Proof. In the previous proposition we constructed u such that if $x \sim y$ then $\hat{u}(x) = \hat{u}(y)$. It is enough now to show that $y \succ x$ if and only if $\hat{u}(y) > \hat{u}(x)$.

Denote by X^i the set of point in $\mathbb{R}^{\mathcal{C}}_+$ such that $x_k = 0$ for all $k \notin \{i, i+1\}$. Clearly, for $x \in X^i$, $\hat{u}(x) \in [u_i, u_{i+1}]$ and $e_{i+1} \succsim x \succsim e_i$. Let $X = \cup_{i=1}^{n-1} X^i$. We first prove the claim for points in X. Suppose $x, y \in X^i$. We can assume that $y_i = x_i$. By the definition of P_{i+1}, $y \succ x$ if and only if $y_{i+1} > x_{i+1}$. But this holds if and only if $\hat{u}(y) \geq \hat{u}(x)$.

Next, suppose that $y \in X^i$ and $x \in X^j$ for $j \neq i$. Then, $y \succ x$ if and only if $i+1 \leq j$ and it is not the case that $i+1 = j$ and $x \sim y \sim e_j$. But this is equivalent to $\hat{u}(y) > \hat{u}(x)$.

Observe now that for every $x \in \mathbb{R}_+^{\mathcal{C}}$ there exists a point $x' \in X$ such that $x' \sim x$. Indeed, there exists i such that $e_{i+1} \succsim x \succsim e_i$. Consider the sets $\mathcal{M}(x) \cap X^i$ and $\mathcal{L}(x) \cap X^i$. By Propositions 4 and 5 these are closed cones. The first contains e_i and the second e_{i+1}. Therefore there exist x' in X^i which belong to both. Thus $x' \sim x$. Now, $x \succ y$ if and only if $x' \sim y'$, which is equivalent to $\hat{u}(x') > \hat{u}(y')$. But, $\hat{u}(x') = \hat{u}(x)$ and $\hat{u}(y') = \hat{u}(y)$, which completes the proof. $\square$

4.9 Proofs of Theorems 1-4

To complete the proof of Theorem 1* we define a probability P on Σ by $P(E) = p\pi(E) = \sum_c p_c P_c(E_c)$. Note, that as $p > 0$, an event E is P-null if and only if $\pi(E) = 0$, which holds, by Proposition 2, if and only if E is null. Now, $\sum_{c_i} u_i P(E \mid C_i) = \hat{u}(\pi(E))$. Since $E \succsim F$ if and only if $\pi(E) \succsim \pi(F)$, (P, u) represents $\succsim$ on Σ by Proposition 12.

Proof of Theorem 1. To prove the "only if" part of Theorem 1 we construct a new state space $(\hat{\Omega}, \hat{\Sigma})$, a new set of consequences $\hat{\mathbb{C}}$, and a new relation $\hat{\succsim}$ on $\hat{\Sigma}$. The set $\hat{\Omega}$ is obtained by eliminating from Ω all events C_i that are null. The σ-algebra $\hat{\Sigma}$ consists of the events in Σ which are subsets of $\hat{\Omega}$. For $\hat{\mathcal{C}}$, we partition the set of consequence for which C_i is non-null into equivalence classes such that c_i and c_j belong to the same class if $C_i \sim C_j$. The consequences in $\hat{\mathcal{C}}$ are these equivalence classes.

We need to show that $\hat{\mathcal{C}}$ has at least two points, that is that there are i and j such that C_i and C_j are non-null and $C_i \succ C_j$.

Let I be the set of indices i such that C_i is non-null. The set I is not empty, because otherwise, $\Omega = \cup_i C_i$ is null, and hence all events are null, contrary to Non-degeneracy. Suppose that all the events C_i with $i \in I$ are similar. Let E be a non-null event. For each $i \notin I$, E_{c_i} is null, and hence, $E \sim \cup_{i \in I} E_{c_i}$. For some indices $i \in I$, E_{c_i} must be non-null. Let I^* be the subset of I of such indices. Then, $E \sim \cup_{i \in I^*} E_{c_i}$. Choose $i^* \in I^*$. Then by Corollary 3, $E \sim E_{c_{i^*}}$. By axiom Int 5 of Consequence Events, $E \sim C_{i^*}$. Since this holds for all non-null events E, and all the C_{i^*} are similar, all non-null events are similar, contrary to Non-degeneracy.

Finally, the relation $\hat{\succsim}$ is the restriction of $\succsim$ to the events in $\hat{\Sigma}$. We skip the simple proof that $\hat{\succsim}$ satisfies axioms Int 1–Int 7 as well as Assumptions 1 and 2. By Theorem 1* there exists a pair $(\hat{P}, \hat{u})$ that represents $\hat{\succsim}$. We define a probability P on Σ by setting $P(E) = \hat{P}(E \cap \hat{\Omega})$. The utility u is defined arbitrarily on c_i that correspond to null C_i, and for all other c_i, $u(c_i) = \hat{u}(\hat{c}_j)$ where $\hat{c}_j$ is the equivalence class of c_i. We omit the straightforward proof that (P, u) represents $\succsim$. □

Proof of Theorem 2. Assume that $\succsim$ satisfies the said properties and (P, u) represents $\succsim$. We show that the conditional probability $P(\cdot \mid C)$ represents the qualitative probability relation $\gtrapprox$ in Definition 4. Since, by Proposition 1 there exists a unique probability on Σ_c that represents this relation, it follows that the conditional parts of probabilities in $\mathcal{P}(\succsim)$ are the same.

Consider an event $A \subseteq C$ and event H such that $H \cap C = \emptyset$. Then, the expected utility given $A \cup H$ is

$$\frac{P(C)P(A \mid C)u_c + \sum_{c' \neq c} P(C')P(H \mid C')u_{c'}}{P(C)P(A \mid C) + \sum_{c' \neq c} P(C')P(H \mid C')}. \tag{8}$$

Choose H such that $C \succ H$ (if there is none, we choose H such that $H \succ C$ and the argument is similar). Then u_c is greater than the expected utility given H. It follows that the derivative of (8) with respect to $P(A \mid C)$ is positive. Thus, For $A, B \subseteq C$, $A \cup H \succsim B \cup H$, which is equivalent to $A \gtrapprox B$, holds if and only if $P(A \mid C) \geq P(B \mid C)$.

A probability vector p is a consequential part of some $P \in \mathcal{P}(\succsim)$ if and only if it is a positive solution of the $n-2$ equations in (4). The set of positive solutions of these equations in the simplex form a maximal interval. dividing equation (4) by p_i we obtain for $i = 2, \ldots, n$,

$$r_i = \frac{1 - \delta_i / r_{i-1}}{\eta_i}, \tag{9}$$

where $r = \rho(p)$. The function $(1 - \delta_i/x)/\eta_i$ is monotonic in $x > 0$. Thus, if q is in the said interval, $s = \rho(q)$, and $r_1 > s_1$, then $r_2 > s_2$, which implies that $r_3 > s_3$ and so on. That is, $p \gg q$. It is easy to check that the maximal interval that contains p and q is ordered.

Conversely, suppose that a family of probability $\mathcal{P}$ satisfies the two properties of the theorem. Let (P_i) be the unique conditional part of probabilities in $\mathcal{P}$. Let $p \neq q$ be two elements in the interval of consequential probabilities of $\mathcal{P}$, such that $q \gg p$. Consider the two equations $\lambda_i p_{i-1} + \eta_i p_{i+1} = p_i$ and $\lambda_i q_{i-1} + \eta_i q_{i+1} = q_i$ with variables δ_i and η_i. It is easy to see that these two equations have a unique solution and that it is positive. We define now a monotonic vector u by equation (7). The vectors p and u satisfy equations (4) and (5). Let $P = \sum p_i P_i$ and let $\succsim$ be the desirability relation defined by the pair (P, u). Then, equations (2) and (3) are satisfied and thus, the set of consequential probabilities of $\mathcal{P}(\succsim)$ is the set of positive solutions of equation (4). Since q is also in this set, $\mathcal{P} = \mathcal{P}(\succsim)$. □

Proof of Theorems 3 and 4. equation (7) shows that $\Delta u_{i+1}/\Delta u_i$ is uniquely determined by the consequential probability vector $(p_i) = (P(C_i))$, which means that u is determined up to a positive affine transformation. Moreover, it satisfies the equation in Theorem 4. □

4.10 Proof of Theorem 5

Proposition 13. *There exists a unique probability measure P on (Ω, Σ) such that for each $\succsim^f \in \mathcal{D}$ and c_i, $P^f(\cdot \mid C_i^f) = P(\cdot \mid C_i^f)$.*

Proof. Let $\mathcal{E}^+$ be the set of non-null events with non-null complements. Let $H \in \mathcal{E}^+$, and f and g be acts such that $H = C_i^f$ and $H = C_j^g$. Let $A, B \subseteq H$. By axiom Int 5 of Consequence Events, $A \sim^f B$ and $A \sim^g B$. Thus, by axiom Ea 5 of Common Likelihood, $\gtrapprox_i^f = \gtrapprox_j^g$ which implies that $P_{c_i}^f = P_{c_j}^g$. We denote this probability on H, which is independent on the consequence, and the act by P_H.

Let $H \subseteq G$ be events in $\mathcal{E}^+$. We show that $P_H(\cdot) = P_G(\cdot \mid H)$. Let f be an act such that $H = C_i^f$ and g an act such that $G = C_i^g$. For $A, B \subseteq H$, again apply axiom Int 5 of Consequence Events and axiom Ea 5 of Common Likelihood to conclude that $A \gtrapprox_i^f B$ if and only in $A \gtrapprox_i^g B$, which means that $P_G(A) \geq P_G(B)$ if and only if $P_H(A) \geq P_H(B)$. But this means that $P_H(\cdot) = P_G(\cdot \mid H)$.

We complete the proof by showing that there exists a unique non-atomic probability P on Σ such that for each $H \in \mathcal{E}^+$, $P_H(\cdot) = P(\cdot \mid H)$.

Let (A, B, X) be a partition of Ω into three non-null events. Then the three events, and the union of each two of them, are all in $\mathcal{E}^+$. We show that $P_{A \cup X}$ and $P_{B \cup X}$ determine P_H for each $H \in \mathcal{E}^+$. Obviously, $P_{H \cap A}(\cdot) = P_{A \cup X}(\cdot \mid H \cap A)$, $P_{H \cap B}(\cdot) = P_{B \cup X}(\cdot \mid H \cap B)$, and $P_{H \cap X}(\cdot) = P_{A \cup X}(\cdot \mid H \cap X) = P_{B \cup X}(\cdot \mid H \cap X)$. It remains to show that $P_{A \cup X}$ and $P_{B \cup X}$ determine $P_H(H \cap A) = \alpha$, $P_H(H \cap B) = \beta$, and $P_H(H \cap X) = 1 - (\alpha + \beta)$. Let $P_{A \cup X}(H \cap A) = p$ and $P_{B \cup X}(H \cap B) = q$. Assume first that $H \cap X$ is non-null. In this case, $p < 1$ and $q < 1$. Then $\alpha/(1 - (\alpha + \beta)) = p/(1 - p)$ and $\beta/(1 - (\alpha + \beta)) = q/(1 - q)$. These two equations determine $\alpha = (p - pq)/(1 - pq)$ and $\beta = (q - pq)/(1 - pq)$. If $H \cap X$ is null, then let $X' \subset X$ be a non-null event such that $X \setminus X'$ is also non-null. Thus,

$H \cup X' \in \mathcal{E}^+$. Now, $P_{H \cup X'}$ is determined, and as $H \subseteq H \cup X'$, P_H is determined.

It is enough now to show that there exists a probability P such that $P_{A\cup X}(\cdot) = P(\cdot \mid A \cup X)$, and $P_{B\cup X}(\cdot) = P(\cdot \mid B \cup X)$. Denote $p = P_{A\cup X}(A)$ and $q = P_{B\cup X}(B)$. Let $P = \alpha P_A + \beta P_B + (1-(\alpha+\beta))P_X$. Then, for the conditionals of P on $A \cup X$ and $B \cup X$ to be as desired, α and β should satisfy the same equations as above, which have indeed a unique solution. □

Proposition 14. *The probability P is the unique element in $\cap_{f\in\mathcal{F}}\mathcal{P}(\succsim^f)$.*

Proof. Let $P^f \in \mathcal{P}(\succsim^f)$. Thus, for each E, $P^f(E) = \sum_i p_i P^f(E \mid C_i^f)$, for some consequential probability vector (p_i). By Proposition 13, $P^f(E) = \sum_i p_i P(E \mid C_i^f)$. We need to show that necessarily $p_i = P(C_i^f)$ and thus $P^f(E) = P(E)$, as required.

Let $g \in \mathcal{F}$ be such that for each i and j, $P(C_i^f \cap C_j^g) > 0$. Then, $P^g(E) = \sum_i q_i P(E \mid C_i^g)$, for some consequential probability vector (q_i). If $P^f = P^g$, then for each j and k, $\sum_i p_i P^f(C_j^f \cap C_k^g \mid C_i^f) = \sum_i q_i P^g(C_j^f \cap C_k^g \mid C_i^g)$. These n^2 equations plus the equations $\sum_i p_i = \sum_i q_i = 1$ as equations in the $2n$ variables (p_i) and (q_i) are independent and hence can have at most one solution. Since $p_i = P(C_i^f)$ and $q_i = P(C_i^g)$ solve these equations, they are the unique solution. □

We say that acts f and g *overlap* if for each i, $C_i^f \cap C_i^g$ is a non-null event.

Lemma 8. *For all acts f and g in $\mathcal{F}$ there exists an act h in $\mathcal{F}$ such that f and h overlap and g and h overlap.*

Proof. By the non-atomicity of the measures P we can choose for each i and j, a partition of $C_i^f \cap C_j^g$ into two events $E_{i,j}$ and $F_{i,j}$ of equal probability. Let $E_i = \cup_j E_{i,j}$ and $F_j = \cup_i F_{i,j}$. Since $P(E_i) = (1/2)P(C_i^f) > 0$, E_i is non-null, and similarly F_j is non-null. Thus the act h defined by $C_k^h = E_k \cup F_k$ is in $\mathcal{F}$. Hence, for each i, $E_i \subseteq C_i^f \cap C_i^h$, f and h overlap. Similarly, g and h overlap. □

Proposition 15. *For each $f, g \in \mathcal{F}$ and i and j, $C_i^f \succsim^f C_j^f$ if and only if $C_i^g \succsim^g C_j^g$.*

Proof. Assume first that f and g overlap. By axiom Int 5 of Consequence Events, $C_i^f \succsim^f C_j^f$ if and only if $C_i^f \cap C_i^g \succsim^f C_i^f \cap C_i^f$. By axiom Ea 4 of Common Desirability this relation holds if and only if $C_i^f \cap C_i^g \succsim^g C_i^f \cap C_i^f$, which, again, by axiom Int 5 of Common Consequences, holds if and only if $C_i^g \succsim C_j^g$. By Lemma 8, the claim holds also for non-overlapping acts. □

By Proposition 15 we can assume without lose of generality that for all f,

$$C_n^f \succsim^f C_{n-1}^f \succsim^f \cdots \succsim^f C_1^f, \tag{10}$$

By Lemma 14 there exists a unique P that belongs to all $\mathcal{P}(\succsim^f)$ for all $f \in \mathcal{F}$. For each such f there exists a unique utility u^f (determined up to a positive affine transformation) such that (P, u^f) represent $\succsim^f$. We now show that the same u serves for all f.

Proposition 16. *There exists a utility vector u which is determined up to a positive affine transformation, such that (P, u) represents $\succsim^f$ for each $f \in \mathcal{F}$.*

Proof. We assume that the desirability relations in equation (10) are strict. The argument for the general case is similar to the one used in the proof of Theorem 1.

We fix $f \in \mathcal{F}$ and omit all superscripts referring to this act. By Proposition 13, $P \in \mathcal{P}(\succsim^f)$ and thus, $\pi(E) = (P(E \mid C_i)) = (P(E \cap C_i)/P(C_i))$ and the consequential part of P is $p = (P(C_i))$. Let x and y be in $\mathbb{R}(i)$ such that $x = \pi(E)$, $y = \pi(F)$, $x \sim y$ and $x \sim^* y$. By equations (2) and (3),

$$\delta_i = \frac{y_{i-1} - x_{i-1}}{x_i - y_i} = \frac{P(F \cap C_{i-1}) - P(E \cap C_{i-1})}{P(E \cap C_i) - P(F \cap C_i)} \frac{P(C_i)}{P(C_{i-1})}, \tag{11}$$

$$\eta_i = \frac{y_{i+1} - x_{i+1}}{x_i - y_i} = \frac{P(F \cap C_{i+1}) - P(E \cap C_{i+1})}{P(E \cap C_i) - P(F \cap C_i)} \frac{P(C_i)}{P(C_{i+1})}. \tag{12}$$

By equation (7), the vector u is determined by,

$$\frac{\Delta u_{i+1}}{\Delta u_i} = \frac{\delta_i p_{i-1}}{\eta_i p_{i+1}} = \frac{P(F \cap C_{i-1}) - P(E \cap C_{i-1})}{P(E \cap C_{i+1}) - P(F \cap C_{i+1})}. \tag{13}$$

Suppose that f and g overlap. Then, it is possible to choose the events E and F in $\cup_i C_i^f \cap C_i^g$. For such events, $E \cap C_j^f = E \cap C_j^g$ for all j and similarly for F. Thus, when we compute the utility vector of g the right-hand side of equation (13) is the same for g and f. We conclude that for the same utility u, (P, u) represents both $\succsim^f$ and $\succsim^g$. By Lemma 8, (P, u) represent all $\succsim^f$ for all f.

□

5 An example

We discuss here the desirability relation $\succsim$ from Examples 1, which is represented by the pair (P, u) where P is the uniform probability distribution on the unit interval. We construct the family of all the probability-utility pairs that represent $\succsim$. We ch0ose an example with three consequences because the case of two consequences is trivial. In this case all of $\Delta(\mathbb{C})$ is an interval ordered by optimism, and all utility functions are positive affine transformation of each other (with Assumption 1).

We project the P-non-null events in Σ to $\mathbb{R}^3_+$, the non-negative orthant of $\mathbb{R}^3$ without 0, by $\pi(E) = (P(E \mid C_i))_i$. By inequality (1) in Definition 3, the desirability relation between two events E and F depends only on $\pi(E)$ and $\pi(F)$. Moreover, if $\pi(E)$ and $\pi(F)$ are proportional then $E \sim F$. This last property makes it possible, just for convenience, to extend $\succsim$ to *all* of $\mathbb{R}^3_+$.

These claims on the relation $\succsim$ on $\mathbb{R}^3_+$ follow easily from the fact that he relation is defined by a probability-utility pair by inequality (1). In the proof of Theorem 1 we need to show that they follow from the axioms.

For $x \in \mathbb{R}^3_+$, let δ and η be the increase in x_1 and x_3 respectively, per a decrease of one unit of x_2, required for maintaining the same probability and the same conditional expected utility. Recalling that $P(C_i) = 1/3$ for $i = 1, 2, 3$, δ and η should satisfy:

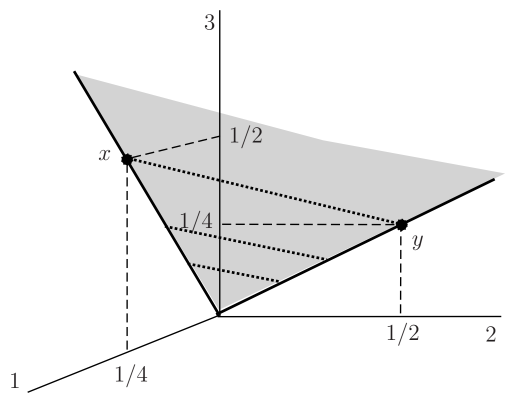
$$(1/3)\delta + (1/3)\eta = (1/3)(1), \tag{14}$$

$$(1/3)(0)\delta + (1/3)(1)\eta = (1/3)(1)(1/2). \tag{15}$$

Equation (14) reflects the preservation of probability. Since, the probability is kept fixed, equation (15) reflects that preservation of the *conditional* expected utility. Observe also, that these equations are the same for all x.

Equations (14) and (15) are derived from the given pair (P, u). In the proof of Theorem 1 we show how they can be derived from the axioms on $\succsim$.

The solution of (14) and (15) is $\delta = \eta = 1/2$. Thus, if the difference $x-y$ of two points x and y in $\mathbb{R}^3_+$ is in the direction $(1/2, -1, 1/2)$, the two points are similar, that is $x \sim y$ and have the same probability, that is $\sum_i (1/3)x_i = \sum_i (1/3)y_i$. In Figure 1, the difference between $x = (1/4, 0, 1/2)$ and $y = (0, 1/2, 1/4)$ is in this direction. Therefore, the whole interval between x and y consists of points which are similar and have the same probability. By the homogeneity of similarity, the cone generated by x and y consists of similar points, and all the points in an interval parallel to the interval $[x, y]$ in this cone have the same probability.



The cone generated by x and y consists of similar points. Each doted line consists of points which as well as being similar have also the same probability.

Figure 1: Similarity and same probability

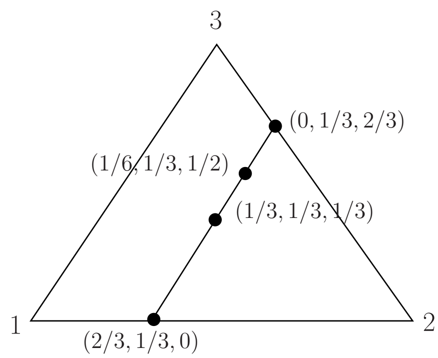
We now show the other pairs (Q, v) that represent the same relation $\succsim$. First, we know by Theorem 2 that $Q(\cdot \mid C_i) = P(\cdot \mid C_i)$ for each i. Thus, the projection of the Q-non-null events to $\mathbb{R}^3_+$ is the same as the projection of the P-non-null events. Also, since (P, u) and (Q, v) present the same desirability relation, the relation $\succsim$ on $\mathbb{R}^3_+$ is the same for both representations.

We show in the proof that having the same probability for two events that are similar is defined in terms of the desirability relation using axiom Int 7 of Persistency. Since (Q, v) and (P, u) represent the same desirability relation, the picture of similarity and having the same probability for (Q, v) should look the same as the one in Figure 1. Thus, the direction of having similarity and same probability should be $(1/2, -1, 1/2)$. Hence, the vector of consequential probability $q = (Q(C_i))_i$ and v should satisfy the following equations:

$$q_1(1/2) + q_3(1/2) = q_3(1), \tag{16}$$

$$q_1 v_1(1/2) + q_3 v_3(1/2) = q_2 v_2(1). \tag{17}$$

The positive probabilities that solve (16) form an open interval of probabilities between $(2/3, 1/3, 0)$ and $(0, 1/3, 2/3)$ as in Figure 2. The point $(1/3, 1/3, 1/3)$ with which we started is, of course, on this line. The closer the point in this interval is to $(0, 1, 3, 2/3)$ the more optimistic it is. Thus, the likelihood ratio vector for $(1/3, 1/3, 1/3)$ is $(1, 1)$, while for $(1/6, 1/3, 1/2)$ it is $(2, 3/2)$ which dominates the first vector.



The closer the point is to (0, 1/3, 2/3) the more optimistic it is.

Figure 2: The interval of consequential probabilities

Fixing q that solves (16) and solving for v in (17) we find that $(v_3 - v_2)/(v_2 - v_1) = q_1/q_3 = (q_1/q_2)(q_2/q_3)$, which is the equality in Theorem 4. Thus, v is uniquely determined by q, up to a positive affine transformation. Moreover, if q is more optimistic than p then the ratio of utility gains of v is dominated by that of u.

6 Concluding comments

A reader who even skimmed through the previous sections would be aware of some obvious or less obvious open questions and possible continuations of this work. We mention here just two of them.

First, the idea of comprehensive state spaces, as introduced by Aumann (1987), is one of the ideas that inspired our work. However, we have dealt only with comprehensive state spaces for a single individual. The next step should be the derivation of probability and utility in multi-agent comprehensive state spaces, and the study of rationality and equilibrium, as in Aumann (1987), in such spaces.

Second, we followed here the orthodoxy of the theory of decisions that considers binary choices as observable (in principle). Modern research in psychology and brain sciences redefines the concept of observability. It includes reportable statements, expressing for example desirability, but excludes utility and probability and gives up observable (in principle) binary choices of acts. Bridging between desirability as discussed here and this modern research calls for a different type of work.

References

Aumann, R. (1987), Correlated Equilibrium as an Expression of Bayesian Rationality, *Econometrica*, 55, pp. 1–18.

Bolker, E. D. (1967), A Simultaneous Axiomatization of Utility and Subjective Probability, *Philosophy of Science*, 34(4), 333–340.

Broome, J. R. (1990), Bolker-Jeffrey Expected Utility Theory and Axiomatic Utilitarianism, *Review of Economic Studies*, 57, 477-502.

de Finetti, B. (1931), Sul significato soggettivo della probabilita, *Fundamenta Mathematicae*, 17, 298–329.

de Finetti, B. (1937), La prèvision: ses lois logiques, ses sources subjectives, *Ann. Inst. H. Poincarè*, 7, 1–68.

Jeffrey, R. (1965), *The Logic of Decision*, McGraw Hill, New York, 1965.

Jeffrey, R. (1983), *The Logic of Decision*, The University of Chicago Press, Chicago, 1983.

Luce, R. D. and D. H. Krantz, (1971), Conditional expected utility, *Econometrica*, 39, 253–271

Ramsey, F. P. (1926), Truth and Probability in *Studies in Subjective Probability*, ed. H. Kyburg, Jr. and H. Smokler, Wiley, New York, 1964.

Savage, L. J. (1954), *The Foundations of Statistics*, Wiley, New York.

von Neumann, J. and O. Morgenstern, (1953), *Theory of Games and Economic Behavior* , 3rd ed., Princeton University Press, Princeton.