Theoretical note

The exponential learning equation as a function of successful trials results in sigmoid performance

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ABSTRACT

While the exponential learning equation, indicating a gradually diminishing improvement, is one of the standard equations to describe learning, a sigmoid behavior with initially increasing then decreasing improvement has also been suggested. Here we show that the sigmoid behavior is mathematically derived from the standard exponential equation when the independent variable of the equation is restricted to the successful trials alone. It is suggested that for tasks promoting success-based learning, performance is better described by the derived sigmoid curve.

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1. Introduction

The exponential learning equation has been derived analytically by several researchers (Estes, 1950; Hull, 1943; Thurstone, 1919) and is one of the standard equations to describe the improvement in the performance of tasks with practice (Heathcote, Brown, & Mewhort, 2000; Ritter & Schooler, 2001):

$$P_n = P_\infty - (P_\infty - P_0) \cdot e^{-\alpha \cdot n},$$

where n denotes trial number, P_n the performance measure at trial n, and p_0, p_∞ the initial and asymptotic performance, respectively and α is a constant rate coefficient. The concavity of P_n implies a monotonically decreasing improvement ($\Delta_n = P_n - P_{n-1} < \Delta_{n-1}$). However, a sigmoid behavior in which the improvement initially increases then decreases has been persistently suggested based on either empirical observations (Culler, 1928; Culler & Girden, 1951; Gallistel, Fairhurst, & Balsam, 2004; Woodworth, 1938) or on analytical derivation from assumptions on the underlying learning process (Gulliksen, 1934, 1953; Mazur & Hastie, 1978; Newell, Liu, & Mayer-Kress, 2001; Thurstone, 1930). Here we note that by an alternative interpretation of the independent parameter n, this presumably contradictory observation of sigmoid behavior can in fact be predicted by the traditional exponential equation.

In conventional usage of the exponential equation, where the independent parameter n equals the number of trials, all trials are assumed to equally affect the learning process. This approach is justified for certain applications of the learning equation. For instance, when describing improvement in the **response time** of well-practiced tasks, such as in the frequently cited cigar-rolling study (Crossman, 1959), it is reasonable to attribute equal weights to all trials. However, applying the equation to describe the improvement in the **success rate** of a task requires a clear distinction between successful and failed trials. When the successful responses are a small fraction of all possible responses, a successful response may provide significantly more information to the learner than a failed response. In the extreme case, when the range of possible responses is very large, a single failed response may provide little (if any) information for improving performance. We therefore propose that for such tasks, the exponential learning equation should be re-defined in terms of the number of successful trials only, rather than the total number of trials, and we show in the next section that this modification leads to the classical sigmoid behavior. In a subsequent section the general case in which the weighted average of success and of failure in facilitating learning is addressed, demonstrating a gradual shift towards a sigmoid function as the weight of successful trials is increased.

2. Success-based learning

Consider a behavioral non-trivial task (i.e. one which requires numerous repeated trials to master) in which each trial is either a success or a failure. In such a case one needs to average the success over a group of trials in order to obtain a non-binary measure of success. If the trial in the standard learning curve is replaced by a block of trials, we obtain

$$P_n = P_\infty - (P_\infty - P_0) \cdot e^{-\alpha \cdot b \cdot n},$$

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n = block number; b = number of trials in each block; p_n = performance measure, defined as the probability of success on a binomial trial in block n.

Practice is assumed to either improve the performance measure or leave it unchanged; therefore, $\forall n \ p_{n+1} \geq p_n$. The above learning equation is modified by replacing the number of trials by the accumulated sum of all previous performances:

$$p_n = p_\infty - (p_\infty - p_0) \cdot e^{-\alpha \cdot s_n}, \tag{1}$$

s_n = accumulated sum of all previous performances until, but not including block n:

$$s_n = \begin{cases} 0 & n = 0 \\ p_0 & n = 1 \\ \sum_{k=0}^{n-1} p_k & n > 1. \end{cases} \tag{2}$$

Note that for every $n \geq 0$, $s_{n+1} = s_n + p_n$.

Rewriting Eq. (1) for $n + 1$,

$$\begin{aligned} p_{n+1} &= p_\infty - (p_\infty - p_0) \cdot e^{-\alpha \cdot s_{n+1}} \\ &= p_\infty - (p_\infty - p_0) \cdot e^{-\alpha \cdot s_n} \cdot e^{-\alpha \cdot p_n}. \end{aligned} \tag{3}$$

Substituting (1) into (3) and rearranging terms, we obtain

$$p_{n+1} = p_\infty - (p_\infty - p_n) \cdot e^{-\alpha \cdot p_n}, \quad n \geq 0. \tag{4}$$

While this first-order nonlinear difference equation can be solved recursively, it is of limited value for analysis purposes. We are unable to solve for its closed form; nevertheless, its continuous analog is tractable, and its closed form solution is the classical sigmoid function.

Define $p(t)$ as

$$p(t) = p_\infty - (p_\infty - p_0) \cdot e^{-\alpha \cdot s(t)}, \quad t \geq 0 \tag{5}$$

$$s(t) = \int_0^t p(x) dx. \tag{6}$$

Eq. (5) is implicit in $p(t)$. An explicit expression is obtained by noting that if $p(t)$ is continuous it is also differentiable and

$$\dot{p} = \alpha p(p_\infty - p), \qquad p(0) = p_0 \tag{7}$$

whose solution is

$$\begin{aligned} p(t) &= \frac{p_0 p_\infty}{p_0 - (p_0 - p_\infty) e^{-\alpha \cdot p_\infty \cdot t}} \\ &= \begin{cases} 0 & p_0 = 0 \\ \frac{p_\infty}{1 + e^{-\alpha \cdot p_\infty \cdot t + C}} & p_0 \neq 0 \end{cases} \end{aligned} \tag{8}$$

$C = \ln \left[(p_\infty - p_0) / p_0 \right].$

The function $p(t)$ in Eq. (8) is the classical sigmoid function with its inflection point at t_i satisfying $p(t_i) = \frac{p_\infty}{2}$. For $0 \leq t < t_i$, $p(t)$ is convex and for $t > t_i$ it is concave. If $p_0 > \frac{p_\infty}{2}$, $p(t)$ is fully concave. Note that without success there is no learning ($p_0 = 0 \Rightarrow p(t) = 0$), which is a reasonable outcome when one asserts success-based learning.

2.1. The general case

When all trials are treated equally, the classical exponential learning function is obtained, while a sigmoid curve is derived when assuming a success-based learning mechanism that restricts learning to successful trials only. In fact the original equation can be generalized to account for both cases. This is done by considering the weighted average of success, $s(t)$, and failure, $F(t)$, $\hat{s}(t) = A \cdot s(t) + (1 - A) \cdot F(t)$. Eqs. (5) and (6) are then replaced by

$$\begin{aligned} \hat{S}(t) &= A \int_0^t p(x) dx + (1 - A) \int_0^t [1 - p(x)] dx \\ &= (1 - A) t + (2A - 1) \int_0^t p(x) dx; \quad 0 \leq A \leq 1 \end{aligned} \tag{9}$$

and

$$p(t) = p_\infty - (p_\infty - p_0) \cdot e^{-\alpha \hat{s}(t)}, \tag{10}$$

where $0 < p_0 \leq p(t) \leq p_\infty < 1$, and $p(t)$ is a monotonically increasing function of t. It can easily be seen that these constraints are reasonable behaviorally. For example, if the initial performance value p_0 is zero and one learns from success trials only, the performance function $p(t)$ will remain zero.

When $A = 0.5$, i.e. success and failure equally affect learning, the learning curve reduces to the standard exponential equation:

$$p(t) = p_\infty - (p_\infty - p_0) \cdot e^{-0.5\alpha \cdot t}. \tag{11}$$

When $A = 1$, learning progresses solely from successful trials, and the learning equation reverts to the original sigmoid equation.

When $A = 0$, $\hat{S}(t) = t - \int_0^t p(x) dx$, $\frac{\dot{p}}{(p - p_\infty)(1 - p)} = -\alpha$ and

$$p(t) = \frac{(p_\infty - p_0) \exp[-\alpha t (1 - p_\infty)] - p_\infty (1 - p_0)}{(p_\infty - p_0) \exp[-\alpha t (1 - p_\infty)] - (1 - p_0)}. \tag{12}$$

When $A \neq 0, 0.5, 1$, $p(t)$ satisfies

$$\frac{\dot{p}(t)}{[p(t) - p_\infty]} = -\alpha \cdot [(1 - A) p_\infty + (2A - 1) p(t)] \quad \text{and}$$

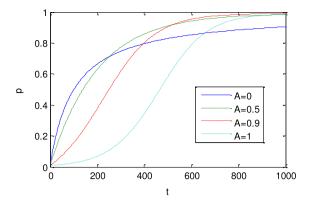$$p(t) = \left\{ C_1 - p_\infty \frac{p_0 - C_1}{p_0 - p_\infty} \exp[-C_2 (C_1 - p_\infty) t] \right\} \bigg/ \left\{ 1 - \frac{p_0 - C_1}{p_0 - p_\infty} \exp[-C_2 (C_1 - p_\infty) t] \right\}, \tag{13}$$

where $C_1 = \frac{(1-A)}{(1-2A)}$; $C_2 = \alpha \cdot (2A - 1)$.

Fig. 1 demonstrates the weighted average-based learning curves. Each curve describes the performance resulting from a different weighted average combination. It can be seen that the learning function exhibits a sigmoid behavior when $A > 0.5$. Furthermore, as more of the learning capacity is assigned to failed trials ('A' decreases, in moving from the rightmost curve towards the left), early stage performance rates increase since failures dominate this stage. However, it is not the optimal strategy since as learning progresses the successful trials dominate the results. Conversely, as more weight is assigned to successful trials ('A' increases, in moving from the leftmost curve towards the right), early stage performance rates decrease, but the later phase of the learning process, dominated by successful trials, is characterized by higher performance rates. The continuous solution $p(t)$ and the recursively calculated p_n were found to compare very well, e.g., when we tried to include in Fig. 1 both solutions the lines simply overlap.

3. Discussion

Sigmoid performance is generally considered contradictory to the classical exponential performance and has been explained either as a noise artifact (Woodworth, 1938), manifestation of a non-exponential learning rule (Culler & Girden, 1951; Gallistel et al., 2004), improving on task-irrelevant features (i.e. understanding the task or the apparatus being used) (Frank Ritter, personal communication) or generated by multiple learning mechanisms progressing either in parallel or in stages (Brooks,

Fig. 1. The learning equation for $A = 0$ (learning from failures only), $A = 0.5$ (learning equally from success and failure − the classical exponential curve), $A = 0.9$, and $A = 1$ (learning only from successful trials, the sigmoid curve). Curves follow an exponential behavior when $A = 0.5$ and sigmoid when $A > 0.5$. As A decreases, (from right most curve towards the left), the early performance rates increase, but the late performance rates decrease. The figure was generated for $p_0 = 0.01$, $p_\infty = 0.99$, $\alpha = 0.01$.

Hilperath, Brooks, Ross, & Freund, 1995; Newell et al., 2001). Here we noted that for some tasks, a sigmoid performance might be simply an extension of the exponential learning process. We reasoned that for success-based tasks, the learning curve equation should be restricted to successful trials alone and showed that this modification to the traditional learning curve gives rise to sigmoid performance.

We suggest that the sigmoid function should be preferred when modeling and fitting data of success-based tasks. Our analysis suggests that this preference becomes more pronounced for tasks with smaller initial success values. Intuitively, for tasks with initial performance values above the critical value $\frac{p_\infty}{2}$, the performance evolution is expected to be entirely concave and describable by an exponential curve. As the performance at early stages of the learning task is reduced, the convex learning patterns become more dominant and the exponential learning patterns become less prominent.

Is learning success-based or failure-based? Let us consider learning in terms of a controller that uses information from previous trials for invoking a learning mechanism, and ask what type of information should be fed to the controller: successful or failed trials? In typical tasks, the stimulus is associated with a single successful and multiple failed responses. Thus, feeding successful trials to the controller would provide most of the information required for learning, advocating success-based learning. However failure-based learning can be advocated from a different perspective: as long as trials end successfully, there is little incentive for modification. In contrast when failed responses occur, it is crucial to feed those trials to the controller in order to learn and avoid them in the future. This latter learning system progresses only through failed responses resulting in failure-based learning. These two opposing approaches to learning may be compared to the distinction between the theory of acquired distinctiveness (Mackintosh, 1975) and the Pearce–Hall model (Pearce & Hall, 1980).

It should be noted that our observations, related to success-based learning are applicable to several specific cases. They do not contradict the reported findings in Gilovich, Vallone, and Tversky (1985) that trials subsequent to successful trials are not improved, since we limit our observation to the early stages of learning and not to its peak performance level. In other learning models (e.g., the early studies reviewed in Wickens, 1982), improved learning is obtained after erroneous trials (Wills, Lavric, Croft, & Hodgson, 2007). The conditions under which subjects employ success-based learning patterns and demonstrate slow initial learning followed

by rapid increase in performance as predicted by the sigmoid function are an open question for future studies.

Finally, it is interesting to contrast the observed sigmoid behavior of the system to the exponential characteristics of its learning mechanism. Sigmoid behavior can resemble an abrupt all or none behavior, while the exponential function is more gradual. This may have contributed to the preference of the exponential over the sigmoid model to describe learning, as the former is better linked to underlying neurobiological behavior (Thurstone, 1919; and others reviewed in Newell et al., 2001 and Gallistel et al., 2004). Yet, the mathematical relation we have noted,

$$p(t) = p_\infty - (p_\infty - p_0) \cdot e^{-\alpha \cdot s(t)} = \frac{p_\infty}{1 + e^{-\alpha \cdot p_\infty \cdot t + C}}, \tag{14}$$

demonstrates that the often observed steep all or none evolution of learning, which in perceptual tasks has been termed "eureka" and "insight" (Ahissar & Hochstein, 1997; Rubin, Nakayam, & Shapley, 1997), does not necessarily imply a similar all or none learning mechanism (Culler & Girden, 1951; Gallistel et al., 2004), and can be interpreted as a mathematical result of a gradual exponential process.

Acknowledgments

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