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Neural Networks 14 (2001) 1153–1159

Neural Networks

www.elsevier.com/locate/neunet

Neural Networks Letter

Best estimated inverse versus inverse of the best estimator

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Received 23 March 2001; accepted 25 June 2001

Abstract

The construction of a feed-forward controller frequently requires the estimation of an inverse function. Two possible methods to achieve this are: (i) learning the best estimated inverse (BEI), a method that is sometimes referred to as direct inverse learning and (ii) learning the inverse of the best estimator (IBE), a method that is sometimes referred to as indirect inverse learning. We analyze a general control problem, in the presence of noise, and show analytically that these two methods are asymptotically significantly different, even for simple linear non-redundant systems. We further demonstrate that the IBE method is typically superior for control purposes.

Keywords: Neurocontrollers; Neural networks for control; inverse control; Motor control; Direct inverse learning; Indirect inverse learning; Feedforward systems; Adaptive control

1. Introduction

In many control applications, there is a need to estimate an inverse model of the controlled system. The notion of feed-forward adaptive control was suggested as a model for biological motor control in the seventies and is frequently used in contemporary models (Albus, 1975; Inbar & Yafe, 1976; Wolpert & Ghahramani, 2000). A method to identify the inverse of linear systems was presented by Widrow, McCool and Medoff (1979). For non-linear systems, the inverse model can be learned by an artificial neural network (ANN). There are numerous papers on using ANN for control, Narendra and Parthasarathy (1990); Barto (1990); Kawato (1990); Bullock, Grossberg and Guenther (1993); Jordan (1996); Karniel and Inbar (2000), and many references therein. Fig. 1 describes the control problem which is to find the control signal X, such that $F(X)$ will be close to a given desired goal, Y_d. $F(X)$ characterizes the controlled system. The input and output spaces can be scalars, vectors, continuous or discrete signals, or Laplace or Z transform domain functions.

There are several different methods for learning the inverse model of a system from input/output samples. The simplest method to construct such an estimated inverse is by feeding the ANN learning algorithm with the input–output samples in the reverse order: outputs as inputs and inputs as desired outputs. This method minimizes the error in the input domain of the system, and is sometimes referred to in the literature as direct inverse learning. We call the model that is learned by this method the best estimated inverse (BEI) model (Fig. 2a). Another method consists of indirect learning. First, the forward model is estimated (Fig. 2b) and then the inverse of the estimated model is constructed. This method minimizes the error in the output domain, finds the best model, and then inverts this model. We call the model that is learned by this method the inverse of the best estimator (IBE) model.

Narendra and Parthasarathy (1990) mentioned the superiority of the indirect learning scheme (here IBE), since it minimizes the output error. Jordan (1996) clearly demonstrates the erroneous results that might occur using the direct inverse learning (BEI) for redundant systems. However, the direct inverse learning approach still appears in numerous papers and software toolboxes without clear manifestation of its drawbacks, even for the simplest non-redundant control systems.

In this paper we show analytically that these two methods are not equivalent, and that the second method, the IBE, is generally superior for control applications in the presence of noise. In Section 2 we formulate and prove the claim by an example of a static system, and show the difference between the BEI and IBE in that case, then, in Section 3 we analyze and simulate the difference between the methods for a dynamic system. In Section 4 we extend our discussion to other methods of learning inverse control, the feedback

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0893-6080/01/$ - see front matter
PII: S0893-6080(01)00098-3

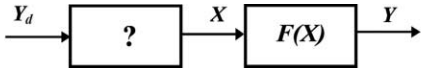

Fig. 1. Feed-forward control and the inverse controller problem.

error learning of Kawato (1990), the distal teacher of Jordan and Rumelhart (1992), and the DIRECT model of Bullock et al. (1993).

2. The claim

Let us phrase the claim in simple words:

Claim: The BEI can be asymptotically significantly different from the IBE.

We actually show that in the presence of noise, with non-zero variance, the two methods differ asymptotically, and the gap increases with the variance of the noise. In the remainder of this section, we phrase the claim formally and prove it by a simple counter-example.

2.1. Formal description of the claim

Consider a system f with input x and output y, and consider a model of the system, $y = \hat{f}(x)$, where $\hat{f} \in F$, a parametric family of models. Suppose that we are given a set of m input/output samples from the system, $\{x_i, y_i\}_{i=1}^{m}$. The x_i are assumed to be independently identically distributed (i.i.d.) according with some given distribution P.

The empirical mean square error between the model and the samples is:

$$D_m(\hat{f}) = (1/m)\sum_{i=1}^{m}\left\|y_i - \hat{f}(x_i)\right\|^2.$$

We will consider a learning algorithm that minimizes the empirical mean square error, and finds the optimal model, $\hat{f}_m^* = \operatorname{argmin}_{\hat{f}}\,[D_m(\hat{f})]$. This type of learning is called a sample error minimization (SEM) algorithm (Anthony & Bartlett, 1999). It is also useful to consider the average square error:

$$D(\hat{f}) = E[\|y - \hat{f}(x)\|^2],$$

according with the given distribution P. We assume that when the number of samples grows to infinity, the empirical error approximates the average error, that is, $\forall \hat{f}$,

$$D_m(\hat{f}) \underset{m\to\infty}{\longrightarrow} D(\hat{f})$$

in some well-defined probabilistic sense (e.g., in probability). With this assumption, known to hold under a wide range of conditions (Anthony & Bartlett 1999), we can use expectation instead of empirical mean for the calculation of the best estimator, $\hat{f}_\infty^*$. The star superscript stands for best and the infinity symbol subscript stands for the asymptotic case, i.e., the number of samples grows to infinity. The basic question at hand is whether the BEI equal the IBE, i.e., does $(\hat{f}^{-1})_\infty^* = (\hat{f}_\infty^*)^{-1}$?

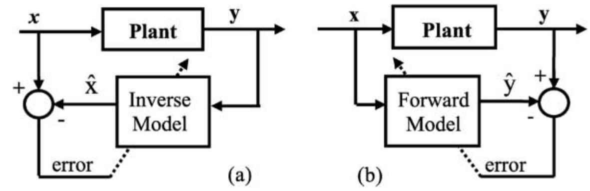

Fig. 2. The two learning methods: (a) The direct inverse model learning scheme that produces the best estimated inverse (BEI). (b) The forward model learning scheme. Inverting the forward model produces the inverse of the best estimator (IBE).

We will show that the answer to this question is negative.

2.2. Proof (by a counter example)

A popular noise model in control problems is an additive noise at the output of the system. Let us examine the system $y = ax + e$. Suppose that we have a series of input/output samples where x_i are independently identically distributed (i.i.d.) random variables, and $y_i = ax_i + e_i$, where e_i are i.i.d. with zero mean and positive (non-zero) variance. The forward model is $\hat{f}(x) = \hat{a}x$, while the inverse model is $[\hat{f}]^{-1}(y) = \hat{b}y$. The inverse of the forward model is: $[\hat{f}]^{-1}(y) = y/\hat{a}$. The question of whether BEI = IBE, is equivalent to: does $\hat{b}_\infty^* = 1/\hat{a}_\infty^*$?

Let us examine the forward model learning error. Since the input samples and the noise are independent, and since the error has zero mean, one can write: $D(\hat{f}) = E[(y - \hat{a}x)^2] = E[(ax + e - \hat{a}x)^2] = E[(-\hat{a})^2x^2] + E(e^2)$. Minimizing this value produces $\hat{a}_\infty^* = a$, therefore the IBE is $1/a$ independent of the noise. In the same vein, let us examine the estimation of the inverse:

$$D\left(\left[\overset{\wedge}{f^{-1}}\right]\right) = E\left[\left(\hat{b}y - x\right)^2\right] = E\left[\left(\hat{b}a - 1\right)^2 x^2\right]$$

$$+ E\left[\left(\hat{b}e\right)^2\right] \Rightarrow \hat{b}_\infty^* = \operatorname*{argmin}_{\hat{b}} D = \frac{aE\left[x^2\right]}{a^2E[x^2] + E[e^2]}$$

.

Therefore $|\hat{b}_\infty^*| < |1/\hat{a}_\infty^*|$, if $E(e^2) > 0$. Note that the difference between the BEI and the IBE is greater when the noise variance is larger, thus, the BEI can be asymptotically significantly different from the IBE.

2.3. Comments

1. The case of additive noise in the input, i.e., $y_i = a(x_i + e_i)$, is equivalent to a change in the noise variance by a factor of a^2, therefore the result will be qualitatively similar.
2. For multiplicative noise, i.e., $y = (a + e)x$, one can show, in the same fashion, that the BEI and the IBE in this case are $a/(a^2 + E[e^2])$ and $1/a$, respectively, therefore again they are different.

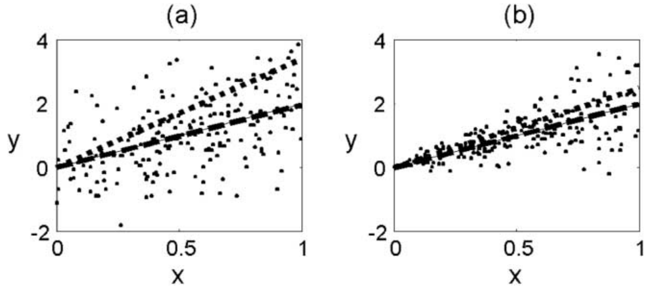

Fig. 3. The IBE model (dashed) versus the BEI model (dotted) calculated from noisy data (dots) of a linear system (solid, thin, almost coincides with the dashed line). (a) Additive noise, (b) multiplicative noise. The IBE and BEI differ in both cases, and the IBE provides a better estimate in both cases.

2.4. Simulation of the counter example

In order to demonstrate the significant difference between the BEI and the IBE, and the proof above, we present a simulation of a simple one-dimensional system.

We have drawn 200 samples of x_i uniformly distributed in the range [0,1], followed by a draw of the noise signal e_i from a normal distribution with zero mean and unit variance. The output samples were $y_i = ax_i + e_i$, where the system parameter was chosen to be $a = 2$. The estimation was performed and the results for one sample path were $\hat{a}^* = 1.933$, $\hat{b}^* = 0.298$. One can see that $\hat{b}^* < 1/\hat{a}^*$. The results of this simulation are plotted in Fig. 3a. One can easily notice the significant difference between the BEI (dotted) and the IBE (dashed). The inverse of the IBE is much closer to the original system (solid).

The same simulation was repeated for multiplicative noise and the result is qualitatively the same (Fig. 3b). For one sample path, the estimated parameters were $\hat{a}^* = 1.965$, $\hat{b}^* = 0.409$. For control purposes the aim is usually to follow a desired output, i.e., to minimize the error in the y-axis of Fig. 3, therefore in the design of a feed forward inverse controller of a static system it is clear that the IBE is superior.

3. A simple dynamical system control

In this section we analyze and demonstrate the BEI and the IBE for a simple first order dynamic system (this system appears in Jordan, 1996). Consider the following plant:

$$x(n+1) = c \cdot x(n) + d \cdot u(n),$$

$$y(n) = x(n) + e(n)$$

where $u(n)$ is the control input, $y(n)$ is the output, $x(n)$ is a state variable and $e(n)$ is an additive output noise. Consider a controller that uses the desired output and an estimator of the state in order to choose the control signal:

$$u(n) = w_1 \hat{x}(n) + w_2 y_d(n+1).$$

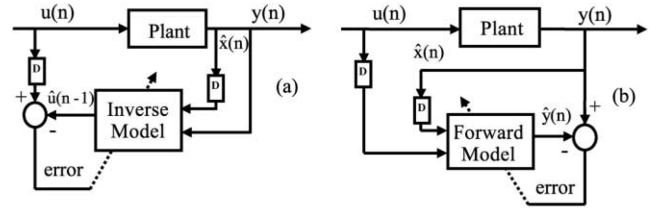

Fig. 4. Inverse model learning (a) and forward model learning (b).

For this system one can estimate the state as being the output and for a feed-forward controller, one can use the desired output:

$$u(n) = w_1 y_d(n) + w_2 y_d(n+1)$$

.

The overall system without the noise, i.e., for $e(n) \equiv 0$ is:

$$y(n+1) = c \cdot y(n) + d \cdot [w_1 y_d(n) + w_2 y_d(n+1)]$$

For a perfect inverse controller, we demand $y(n) = y_d(n)$ and calculate the perfect control parameters:

$$[c + dw_1]y(n) + [dw_2 - 1]y(n+1) = 0$$

$$\Rightarrow w_1 = -c/d,\ w_2 = 1/d.$$

In practice c and d are unknown and $e(n) \neq 0$. In this setup, let us compare the IBE and the BEI methods to estimate the control parameters. The BEI method finds $\hat{w}_1^{\mathrm{BEI}}, \hat{w}_2^{\mathrm{BEI}}$, that minimizes $D([\hat{f})^{-1}]) = E[(u(n) - \hat{u}(n))^2)$ (Fig. 4a). The IBE method finds $\hat{c}, \hat{d}$ that minimizes $D(\hat{f}) = E[(y(n) - \hat{y}(n))^2]$ (Fig. 4b) and then calculates $\hat{w}_1^{\mathrm{IBE}} = -\hat{c}/\hat{d}$, $\hat{w}_2^{\mathrm{IBE}} = 1/\hat{d}$.

Proposition. For the system described above, let $u(n)$ and $e(n)$ be i.i.d. random variables, independent of each other, with zero mean and variance $E[u^2]$ and $E[e^2]$ respectively. The estimated parameters of the BEI and IBE when $n \to \infty$ are:

$$\hat{w}_1^{\mathrm{IBE}} = \frac{c \cdot d \cdot E[u^2]}{(c^2 - 1) \cdot E[e^2] - d^2 \cdot E[u^2]}$$

$$\hat{w}_2^{\mathrm{IBE}} = \frac{1}{d}$$

$$\hat{w}_1^{\mathrm{BEI}} = \frac{c \cdot d^3 \cdot (E[u^2])^2}{(c^2 - 1)(E[e^2])^2 - 2d^2 E[e^2]E[u^2] - d^4 (E[u^2])^2}$$

$$\hat{w}_2^{\mathrm{BEI}} = \frac{d \cdot E[u^2]\{(1 - c^2)E[e^2] + d^2 E[u^2]\}}{(1 - c^2)(E[e^2])^2 + 2d^2 E[e^2]E[u^2] + d^4 (E[u^2])^2}$$

Proof. The proof follows the same principles describes in Section 2 for this two-dimensional optimization problem. The details are deferred to the appendix.

Note that without noise, i.e., $E[e^2] = 0$, both methods converge to the 'perfect inverse'. One can also see that in

Table 1
The controller parameters w_1 and w_2 for the system described in the text, calculated and simulated by the BEI and the IBE methods. The calculated data uses the results of the proposition with the noise and input parameters from the simulations. One can see that the parameters produced by the IBE method are closer to the calculated perfect controller

	Perfect controller	BEI calculated	BEI simulated	IBE calculated	IBE simulated
w_1	−1.25	−0.808	−0.821	−1.053	−1.11
w_2	2.5	1.919	1.933	2.500	2.514

terms of the parameter values, the IBE is closer to the perfect parameters than the BEI. If we had a perfect knowledge of the state during the estimation process (Fig. 4) the IBE would have converged to the values of the perfect inverse. However this case is not realistic, therefore we analyzed the case with noise in the state estimation. Still, the IBE is usually superior to the BEI, as we show in the following simulations that demonstrate the consequences of the difference between the methods.

We used a batch of 500 input/output time steps taken from the system described above, where the inputs $u(i)$ are i.i.d., normally distributed with variance 0.5 and zero mean, and $e(n)$ are i.i.d., normally distributed with a variance of 0.1 and zero mean. We calculate the perfect controller as well as the BEI and the IBE controllers for a stable system ($c = -0.5$, $d = 0.4$). The parameters of the various controllers, calculated and simulated for a single run are presented in Table 1.

These three controllers were fed by a desired output consisting of a few steps and ramps and their performance appears in Fig. 5. The dotted line in Fig. 5 represents the perfect controller's performance. It is close but not equal to the solid line since there is noise in the system. One can see that typically the IBE (dashed) is superior to the BEI (dot–dashed).

4. Other methods

In this section, we briefly examine three more sophisticated learning algorithms. The feedback error learning of Kawato (1990), the distal teacher of Jordan and Rumelhart (1992), and the DIRECT model of Bullock et al. (1993). For the first two, we show that in the case of a static linear system with some assumptions, the preferred inverse (i.e., the IBE) can be estimated. For the third model the BEI is used and the possible advantaged for redundant system control are discussed.

4.1. Feedback error learning

In the feedback error learning scheme, there is a simple feedback controller (such as a PID controller) and an ANN that learns the inverse model of the plant by minimizing the motor error. For further details, see Kawato (1990).

Let us analyze the simplest situation where the plant is just a linear amplifier, a, the feedback is proportional feedback k, and the ANN consists of a single parameter w.

$$x = k(y_d - y) + w \cdot y_d$$

$$y = ax + e$$

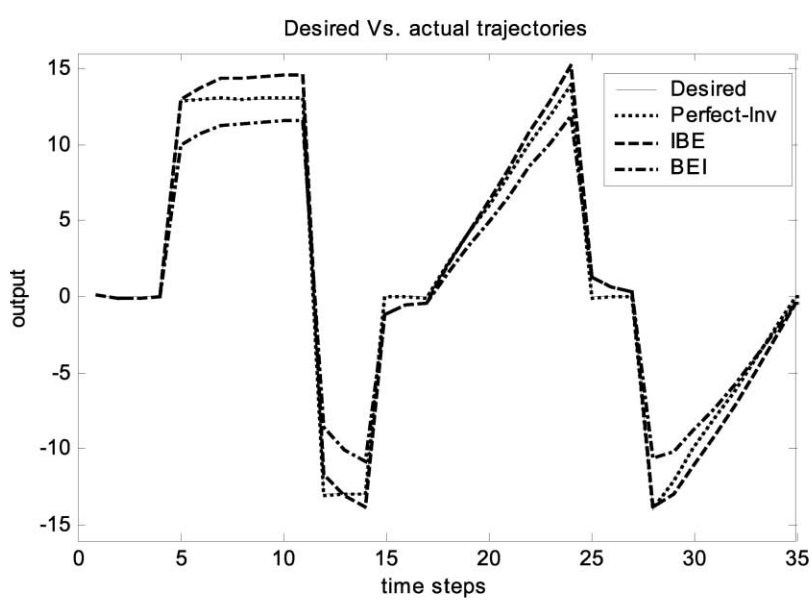

Fig. 5. A first order system control. The IBE controller performance (dashed) versus the BEI controller performance (dot–dashed). The thin solid line (that is almost under the dotted line) is the desired trajectory and the dotted line is the performance of the perfect inverse controller.

Denote the discrepancy measure to be minimized by J, and the motor error by P:

$$J = |P|^2 = |k(y_d - y)|^2$$

$$P = k(y_d - [e + a(P + w \cdot y_d)])$$

$$P = (k \cdot y_d(1 - aw) - ke)/(1 + ka)$$

$$E|P|^2 \to \min \Rightarrow \; w^* = 1/a$$

That is, the inverse model will tend to converge to the IBE.

4.2. Supervised learning with distal teacher

In the case of supervised learning with a distal teacher, there is a forward model that is learned by the prediction error, and an inverse model that is used as the controller and is learned by the performance error that is backpropogated through the forward model. For further details of this method, see Jordan and Rumelhart (1992).

Let us analyze the simplest situation where the plant is just a linear amplifier a, the forward linear model is w_F, and the inverse linear model is w_I, i.e.:

$$x = w_I \cdot y_d$$

$$y = ax + e.$$

$$J(\text{performance}) = |y_d - y|^2 = |y_d - (e + a \cdot w_I \cdot y_d)|^2$$

$$= |(1 - a \cdot w_I) \cdot y_d - e|^2$$

$$EJ \to \min \Rightarrow \; w_I^* = 1/a$$

Here again the inverse controller tends to converge to the IBE.

In this architecture, in order to minimize the performance error, we assume a perfect forward model, i.e. $w_F = a$. This assumption is consistent, since we have proved that it holds asymptotically for the linear case.

4.3. The DIRECT model

The DIRECT model (stands for Direction-to-Rotation Effector Control Transform) estimates the transformation between spatial velocities and effector velocities. For further details, see Bullock et al. (1993).

In robotic control the relation between the spatial velocities $\dot{x}$ and the joint velocities $\dot{\sigma}$ is presented as $\dot{x} = J(\sigma)\dot{\sigma}$, where $J(\sigma)$ is the Jacobian matrix. For redundant systems, a unique inverse for $J(\sigma)$ does not exists. One approach could be to construct the complete IBE, i.e., represent all the possible solutions (Karniel, Meir & Inbar, 2001). A common approach in robotics is to calculate one solution (one of the possible IBE), usually the Moore-Penrose Pseudoinverse. However, these controllers may suffer from large command signals, inconvenient positions, or non-integrable solutions (Mussa-Ivaldi & Hogan, 1991; Guenther & Micci Barreca, 1997). The DIRECT model essentially uses the BEI method for learning. However, combined with the integration and feedback elements it can lead to a controller that is much more well-behaved near singularities (Guenther & Micci Barreca, 1997). The DIRECT method was also shown to be able to deal with redundancy (Bullock et al., 1993), which is a well-documented drawback of the BEI (see Jordan, 1996 where the term 'direct inverse modeling' is used for a system that finds the BEI).

5. Discussion

The essence of the difference between the BEI and the IBE lies in an old linear regression problem of the difference between inverse and direct regression. However, this problem, even in its simplest formulation as a linear regression problem, seems to be a source of everlasting confusion. We traced this debate between methods of learning inverse regression back to 29 December 1938 when Dr C. Eisenhart presented it before the American Statistical Association in Detroit (Eisenhart, 1939). He says there, "*It does not seem to be generally realized that the fitting should be done in term of the deviations which actually represent*'*error*'", and then he presents some problems in linear regression, and in particular the difference between direct and inverse fitting. However, Krutchkoff (1967, 1969) presented some numerical examples for the opposite case. This debate continues with the work of Halperin (1970). We do not wish to go into the details of these papers, however it is clear that the proper method depends on the exact definition of the aim of the fitting. Another related discussion appears in the Econometrics literature (Conway & Roberts, 1983; Goldberger, 1984; Greene, 1984) and a recent analysis appears in Chow and Shao (1990) with an example from the US pharmaceutical industry that used an improper method of estimation.

In contrast to many of the examples above, where the preferred method is not always clear, in control problems we usually aim to reduce the error in the output of the system rather than in the control signal and in this case it is clear that the IBE method is superior to the BEI. The BEI appears frequently in many research works, applications and software simulation toolboxes, starting from the seminal work of Widrow et al. (1979). The field of neural computation introduces many new possible methods for learning control with ANN. It is important to carefully check the performance measure that is being minimized, and consider the difference between the BEI and the IBE.

Acknowledgements

We wish to thank Raphael Sivan for his useful suggestions on an earlier version of this manuscript.

Appendix A. Proof of the proposition of Section 3

Let us start with the IBE, which finds first the best estimation of a forward model (Fig. 4b), i.e., finds the parameters $\hat{c}^*_\infty$, $\hat{d}^*_\infty$, that minimizes the following:

$$D(\hat{f}) = E[(y(n) - \hat{y}(n))^2] = E[(x(n) + e(n) - \hat{c}\cdot y(n-1) - \hat{d}\cdot u(n-1))^2]$$

$$= E[(c\cdot x(n-1) + d\cdot u(n-1) + e(n) - \hat{c}\cdot x(n-1) - \hat{c}\cdot e(n-1) - \hat{d}\cdot u(n-1))^2]$$

Note that $e(n), e(n-1)$, $u(n-1)$, $x(n-1)$ are independent of each other, and since we are analyzing the case of an infinite number of examples we are actually concerned with the case $n\rightarrow\infty$. Therefore:

$$D(\hat{f}) = (c-\hat{c})^2\cdot E[x^2] + (d-\hat{d})^2\cdot E[u^2] + (1+\hat{c}^2)\cdot E[e^2]$$

In this case it is clear that $\hat{d}^*_\infty = d$. To obtain $\hat{c}^*_\infty$ one needs to minimize the term:

$$D(\hat{f}) = (c-\hat{c})^2\cdot E[x^2] + (1+\hat{c}^2)\cdot E[e^2]$$

$$\frac{\partial}{\partial\hat{c}}D(\hat{f}) = 0 \Rightarrow \hat{c}^*_\infty = \frac{c\cdot E[x^2]}{E[x^2]+E[e^2]}$$

From the plant equations:

$$E[x^2] = c^2\cdot E[x^2] + d^2\cdot E[u^2] \Rightarrow E[x^2] = \frac{d^2\cdot E[u^2]}{1-c^2}$$

Therefore:

$$\hat{c}^*_\infty = \frac{c\cdot d^2 E[u^2]}{d^2\cdot E[u^2] + (1-c^2)\cdot E[e^2]}$$

Finally, the IBE parameters are:

$$\hat{w}_1^{\mathrm{IBE}} = -\hat{c}^*_\infty/\hat{d}^*_\infty,$$

$$\hat{w}_2^{\mathrm{IBE}} = 1/\hat{d}^*_\infty$$

$$\hat{w}_1^{\mathrm{IBE}} = \frac{c\cdot d\cdot E[u^2]}{(c^2-1)\cdot E[e^2] - d^2\cdot E[u^2]}$$

$$\hat{w}_2^{\mathrm{IBE}} = 1/d$$

As for the BEI, we wish to find the parameters $\hat{w}_1^{\mathrm{BEI}}$, $\hat{w}_2^{\mathrm{BEI}}$ that minimize the following expression (Fig. 4a):

$$D([\hat{f^{-1}}]) = E[(u(n-1) - \hat{u}(n-1))^2]$$

Using the controller and plant equations one can get:

$$u(n-1) - \hat{u}(n-1) = u(n-1) - \hat{w}_1\cdot y(n-1) - \hat{w}_2\cdot y(n)$$
$$= u(n-1) - \hat{w}_1\cdot c\cdot x(n-2) - \hat{w}_1\cdot d\cdot u(n-2)$$
$$-\hat{w}_1\cdot e(n-1) - \hat{w}_2\cdot c^2\cdot x(n-2) - \hat{w}_2\cdot c\cdot d\cdot u(n-2)$$
$$-\hat{w}_2\cdot d\cdot u(n-1) - \hat{w}_2\cdot e(n)$$

Note also that $e(n)$, $e(n-1)$, $u(n-1)$, $u(n-2)$, $x(n-2)$ are independent of each other, and since we are analyzing the case of infinite number of examples we are actually concerned with the case $n\rightarrow\infty$. Therefore:

$$D([\hat{f^{-1}}]) = E[u(n-1) - \hat{u}(n-1))^2] =$$

$$= (1-\hat{w}_2\cdot d)^2 E[u^2] + (\hat{w}_1\cdot d + \hat{w}_2\cdot c\cdot d)^2 E[u^2] + (\hat{w}_1\cdot c + \hat{w}_2\cdot c^2)^2 E[x^2] + (\hat{w}_1^2 + \hat{w}_2^2)E[e^2]$$

In order to minimize this function we rearrange it to a standard quadratic form as follows:

$$D([\hat{f^{-1}}]) = E[u^2] + (2\cdot c^3 E[x^2] + 2\cdot c\cdot d^2 E[u^2])\hat{w}_1\cdot\hat{w}_2 - 2\cdot d\cdot E[u^2]\cdot\hat{w}_2$$
$$+ (c^2 E[x^2] + E[e^2] + d^2 E[u^2])\hat{w}_1^2 + (c^4 E[x^2] + E[e^2] + (c^2+1)\cdot d^2 E[u^2])\hat{w}_2^2$$
$$= E[u^2] + \tfrac{1}{2}W^T Q W - a^T W$$

where:

$$W = \begin{bmatrix}\hat{w}_1\\ \hat{w}_2\end{bmatrix} a = \begin{bmatrix}0\\ 2\cdot d\cdot E[u^2]\end{bmatrix} Q = \begin{bmatrix}q_{11} & q_{12}\\ q_{21} & q_{22}\end{bmatrix}$$

$$q_{11} = 2\cdot\{c^2 E[x^2] + E[e^2] + d^2 E[u^2]\}$$

$$q_{12} = q_{21} = 2\cdot\{c^3 E[x^2] + c\cdot d^2 E[u^2]\}$$

$$q_{22} = 2\cdot\{c^4 E[x^2] + E[e^2] + (c^2+1)\cdot d^2 E[u^2]\}$$

The minimum of this quadratic function is:

$$W^* = Q^{-1}\cdot a = \frac{1}{q_{22}\cdot q_{11} - q_{21}\cdot q_{12}}\begin{bmatrix}q_{22} & -q_{12}\\ -q_{21} & q_{11}\end{bmatrix}\cdot\begin{bmatrix}0\\ 2\cdot d\cdot E\left[u^2\right]\end{bmatrix}$$

$$= \frac{2\cdot d\cdot E[u^2]}{q_{22}\cdot q_{11} - q_{21}\cdot q_{12}}\begin{bmatrix}-q_{12}\\ q_{11}\end{bmatrix}$$

Note that from the plant equations:

$$E[x^2] = \frac{d^2\cdot E[u^2]}{1-c^2}$$

Substituting all the above into W^* we get:

$$\hat{w}_1^{\mathrm{BEI}} = \frac{c\cdot d^3\cdot(E[u^2])^2}{(c^2-1)(E[e^2])^2 - 2d^2 E[e^2]E[u^2] - d^4(E[u^2])^2}$$

$$\hat{w}_2^{\mathrm{BEI}} = \frac{d\cdot E[u^2]\{(1-c^2)E[e^2] + d^2 E[u^2]\}}{(1-c^2)(E[e^2])^2 + 2d^2 E[e^2]E[u^2] + d^4(E(u^2))^2}$$

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