An active sensing principle for haptic interaction with dynamical systems

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Abstract

What is it inside the colorfully wrapped present? You pick it up to ear level and listen, shake it, and then listen again. To you, the basic principle of active sensing is quite clear – first absorb, then, if there is no movement or sound, shake it and then reabsorb.

We propose an extremely basic hypothesis for the active sensing of haptic interaction with dynamical systems. Our hypothesis asserts that in order to improve the efficiency of extracting information from a probed system, the sensor should act according to the following basic principle: if the probed system is passive, the sensor should be active; conversely, when the probed system is active, the sensor should be passive.

We proved the proposed principle for interaction with a second-order mechanical system with the goal to enhance classification performance between two possible sine power sources. We showed that the addition of an active power source to a passive testing sensor leads to decreased sensitivity in the amplitude and frequency of the tested power source. Further, an extension of this principle is provided, presenting the conditions for reduced sensitivity to spring and damper parameters.

To test its applicability for a linear system in a noisy environment, a computer simulation was performed demonstrating that classification performance improved by following the proposed principle.

Lastly, ten subjects probed an active virtual system under either active or passive conditions. A comparison of the mean just-noticeable difference (jnd) of both conditions indicated significantly better sensitivity was obtained by following the principle.

Keywords*:* Sensitivity analysis, classification, just-noticeable difference, human-machine interaction

1. Introduction

In everyday life, we are bombarded by a rapid and endless stream of information requiring decisions. From the moment we are born and all through our lives, we spend a vast amount of our time exploring our environment. As we grow, we become more and more adept in the process of selecting the appropriate strategy for each “exploration task” we perform: We squeeze a ball to sense its compliance, we let tap water run through our fingers to determine its temperature, and we lift a metal box to estimate its weight.

This exceptional ability we have is commonly known as “Active Sensing.” The Active Sensing paradigm is driven by a very basic logic: given the combination of a vast amount of data and limited processing resources and time, it is fundamental for the data acquisition process to be selective (Bajcsy and Campos 1992).

Over the last three decades, this concept has been largely investigated with studies referring both to the macro level of general definitions and the micro level, describing mission specific algorithms in the fields of active vision, haptic object recognition, and methods of implementation in machine perception.

Bajscy addressed the question of what active sensing is, arguing that this concept refers not only to “active sensors” emitting energy to probe the environment, but also to "passive sensors employed actively" (Bajcsy 1988). The idea was extended through the definition of "exploratory procedures" for haptic object recognition (Lederman and Klatzky 1987), task modeling, system architecture, and algorithms for a task combining both visual and haptic exploration (Bajcsy and Campos 1992).

When discussing active sensing in humans, one relevant research direction is the sensitivity of humans to dynamical systems. Psychophysical experiments have been conducted to explore human sensitivity by measuring the just-noticeable difference (jnd) to different system parameters in various conditions (Pang, Tan et al. 1991; Tan, Pang et al. 1992; Beauregard, Srinivasan et al. 1995; Allin, Matsuoka et al. 2002). For example, Pang et al. (1991) utilized an electromechanical device, which produced a constant resistance force, to estimate the jnd for a manual discrimination of force by active finger motion. A similar setup was utilized by Tan et al. (1995) to explore the effect of perceptual cues on compliance resolution. In a recent study by Israr et al. (2009), subjects were instructed to remain active or passive while interacting with a virtual dynamical system. This system could either be excited by the subject (active subject/passive system) or by an external source (passive subject/active system). However, those studies do not consider the possible influence of the interrogators activity on the sensitivity while interacting with an active dynamical system, as suggested by our active sensing principle.

Whether discussing active sensing in the context of sophisticated computational algorithms or in that of human exploration, it seems there is no simple answer to the following basic question: *When should the probing sensor be active and when should it be passive?*

In this paper, we have addressed the procedure of interaction with a dynamical system, defining "active system" as one containing energy producing elements and focusing on strategy to maximize the sensitivity to a given system parameter. We aimed to initialize a systematic theoretical approach to analyze the interaction between two dynamical systems of a sensor and its probed dynamical environment in order to answer the above question. Thus, we proposed an active sensing principle for haptic interaction with a dynamical system, formulated in the context of a classification problem, and then proved the principle for a basic linear mechanical system using phasor sensitivity analysis. The proof of this logical and basic macro level principle sets the grounds for improved data extraction in any interaction with the subclass of second-order dynamical systems.

This paper is structured as follows: In Section 2, we describe the connection between system sensitivity and classification performance. Section 3 contains the claim and the analytical proof. Section 4 presents a simulation example of a classification process demonstrating the principle in a noisy environment. Section 5 describes a psychophysical experiment with human subjects probing virtual systems. Lastly, a discussion is provided in Section 6.

2. Sensitivity and Classification

Consider the following basic binary classification task: Given an output of a dynamical system, Y, we would like to classify the value of one of its parameters, X, above or below the threshold value X_0. For this purpose, we wish to maximize the difference in the measured output for small changes in the parameter value.

In the presence of white Gaussian noise, the slope of Y with respect to X directly determines our ability to discriminate between two adjacent values of X, indicating the success rate of the best possible classifier. As shown in Figure 2.1, the output of the system depicted in red will clearly lead to a better classification performance than the system depicted in blue.

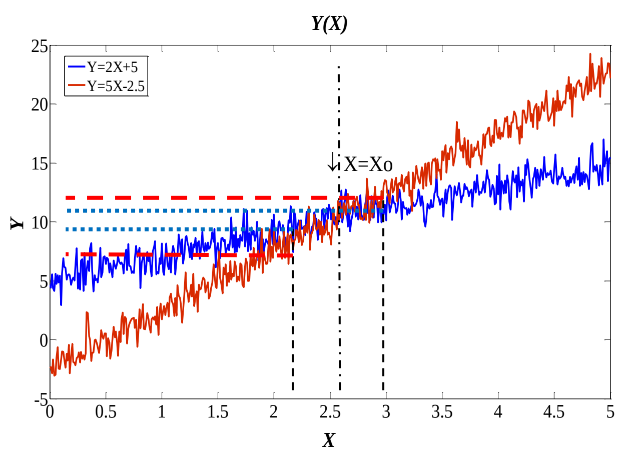

Figure 2.1 Output of a system Y as a function of an inner parameter X (contaminated with white Gaussian noise at 1 dB). Blue and red solid lines represent two possible relations between the system's output and one of its parameters. Dotted and dashed lines represent the changes in the value of output for each of the two relations for similar values of parameter X.

One natural way to analyze this relationship is with the use of “Sensitivity” (S), defined as:

$$(2.1)\ \ S = \left|\frac{\partial \mathrm{Y}}{\partial \mathrm{X}} \cdot \frac{\mathrm{X}}{\mathrm{Y}}\right|$$

The sensitivity depends upon the system's output response to parameter change at the proximity of the operating point. Our underlying assumption is that for a given value $X = X_0$, there is an adjunct environment where the input-output relationship is approximately linear. Hence, to measure the influence of a parameter value on the system's output, we simply derive the output of the system with respect to the parameter at the operating point.

Weber's law addresses the minimal change in stimuli magnitude required to produce an evident alteration, a just-noticeable difference (jnd) in the sensory sensation (Weber 1996). In its classical formulation, the law states that the ratio between the increment for noticeable difference ΔI and the background intensity, known as the Weber Fraction (WF), is constant:

$$(2.2) WF = \frac{\Delta I}{I} = Constant.$$

This concept is closely related to the notion of sensitivity. Let us assume that for an incremental input change of ΔX_0, the change in output value is ΔY_0. If we could enhance the sensitivity at this operating point by a factor of α $(\alpha > 1)$, it would enable us to observe the exact same output change for an incremental input change of $\Delta \hat{X}_0 = \frac{\Delta X_0}{\alpha} < \Delta X_0$. Previously, we were able to detect a change of ΔX_0 in our parameter, whereas now it is possible to detect a change of $\frac{\Delta X_0}{\alpha}$. The direct outcome of this enhanced sensitivity is the decrease in the Weber fraction: $WF_2 = \frac{WF_1}{\alpha}$, enabling improved classification performance. In other words, the discrimination capability could be equally measured by either the success rate or by the WF because sensitivity correlates with both.

Returning to our discussion of active sensing, the question now becomes: *What kind of strategy applied while interacting with a dynamical system will yield the highest possible sensitivity?*

3. The Principle

3.1 Assumptions

The linear second-order mechanical system described in Figure 3.1 consists of a combination of a testing system comprised of a spring, a damper, and an optional power source, in addition to an active tested system comprised of a spring, a damper, a mass, and a sine power source. We considered the Root Mean Square (RMS) of the mass trajectory and examined the system's sensitivity to a change in the frequency and amplitude of the power source. We compared system sensitivity under two testing conditions—with and without an additional power source—and analytically derived that *the additional power source will lead to a reduction in the system's sensitivity* under the following conditions:

1. The whole system consisting of both the tested system as well as the testing input is at a steady state.

2. The force function of the testing system can be presented as a weighted sum of its sinusoidal functions.

3.2 Claim

The introduction of an additional power source to a testing system (right side of Fig. 3.1) will result in a reduction of system sensitivity with respect to a change in the frequency or amplitude of the tested sine power source and will, consequently, degrade the classification performance (Fig. 3.2).

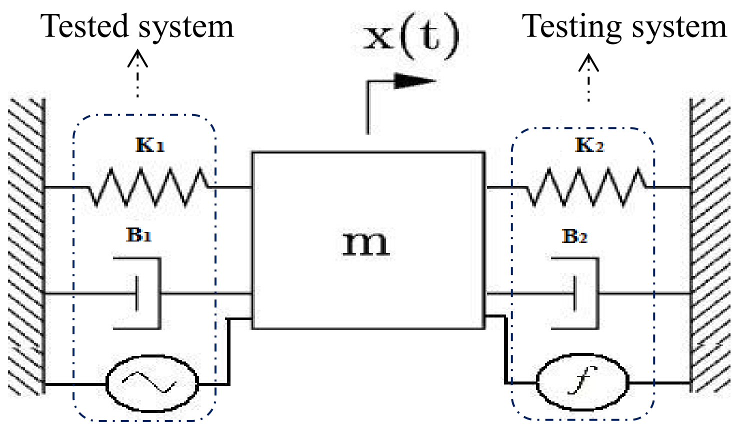

Figure 3.1 Second-order mechanical system consisting of a spring, a damper, a sine power source, and a mass interacting with a spring, a damper, and an optional power source.

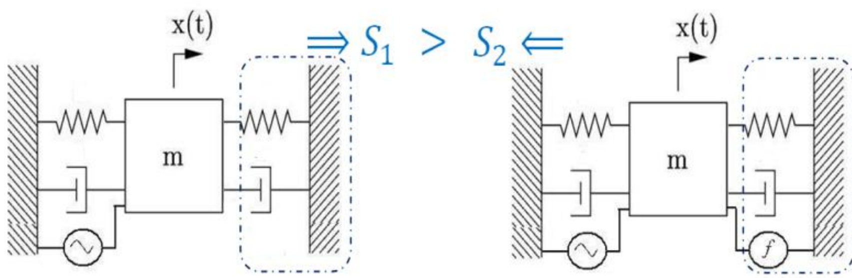

Figure 3.2 Claim: Introduction of an additional power source to a testing system will result in a reduction of system sensitivity.

3.3 Sensitivity to frequency

In this section, we prove the claim for sensitivity to frequency, and in the next section we prove it for sensitivity to amplitude. A detailed analytical calculation can be found in Appendix A, and a summary of the analysis results can be found in Appendix C.

We begin the analysis by considering a single sine testing power source, then we extend it to a sum of sine power sources, and finally we consider the case in which one of the power sources was at the same frequency as the tested power source. We start with a system in which our testing sensor was passive ($f = 0$).

The system's differential equation is given by:

$$(3.3.1)M\ddot{X}_1 + B\dot{X}_1 + KX_1 = Asin(\omega_1 t)$$

Where X_1 is the mass trajectory, $B = (B_1 + B_2)$, $K = K_1 + K_2$, and $\omega_1 = 2\pi f_1$.

This is a classical equation representing a damped-driven harmonic oscillator. In our case, the solution of this second-order inhomogeneous differential equation for $X_1$is a combination of a transient (i.e., homogenous) solution of the form $X_h = C_1 X_{1_1} + C_2 X_{1_2}$ (where X_{1_1} and X_{1_2} are decaying exponents) and a steady state solution of the form $\tilde{C}\sin(\omega_1 + \theta)$ (where $\tilde{C}$ is a constant and θ is the phase shift, in this case 0). Following the assumptions, we address only the steady state solution. Since our focus was on sinusoidal power sources, our steady state assumption enabled us to use phasor analysis, choosing the commonly utilized Root Mean Square (RMS) representation for the mass magnitude rather than the peak of the amplitude sinusoidal.

The RMS of the output is given by:

$$(3.3.2)X_1 = \frac{A}{\sqrt{2\left[\left(K - \omega_1^2 M\right)^2 + \omega_1^2 B^2\right]}}$$

We denote

$$\alpha_1 = (K - \omega_1^2 M)^2 + \omega_1^2 B^2$$

And derive the system's sensitivity:

$$(3.3.3)S_{1_{\omega_1}} = \left|\frac{\partial X_1}{\partial \omega_1} \cdot \frac{\omega_1}{X_1}\right| = \left|\frac{\left(2\omega_1 M\left(K - \omega_1^2 M\right) - \omega_1 B^2\right)}{\alpha} \cdot \omega_1\right|$$

Next, we analyzed the **addition of a weighted sum of sine power source**s.

The updated differential equation is:

$$\mathbf{(3.3.8) M\ddot{X}_1 + B\dot{X}_1 + KX_1 = A\sin(\omega_1 t) + \sum_{i=2}^{N} A_i \sin(\omega_i t)}$$

The RMS is given by:

$$\mathbf{(3.3.9) X_2 == \frac{1}{\sqrt{2}} \sqrt{\sum_{i=1}^{N} \frac{A_i^2}{\left(K - \omega_i^2 M\right)^2 + \omega_i^2 B^2}}}$$

The sensitivity:

$$S_2 = \left|\frac{\partial X_2}{\partial \omega_1} \cdot \frac{\omega_1}{X_2}\right| = \left|\frac{{A_1}^2 \cdot \frac{(2\omega_1 M(K-\omega_1^2 M) - \omega_1 B^2)}{\alpha^2}}{\frac{A_1^2}{\alpha}\left(1 + \sum_{i=2}^{N} \frac{A_i^2 \alpha}{A_1^2 \alpha_i}\right)} \cdot \omega_1\right|$$

Where:

$$\alpha_i = (K - \omega_i^2 M)^2 + \omega_i^2 B^2$$

As before, we notice:

$$\mathbf{(3.3.10) S_2 = \frac{S_{1\omega_1}}{1 + \underbrace{\sum_{i=2}^{N} \frac{A_i^2 \alpha}{A_1^2 \beta_i}}_{>0}} < S_{1\omega_1}}$$

Hence, we concluded that the addition of a weighed sum of sine power sources will lead to a reduced sensitivity to frequency change.

We now address the case in which the expression $\sum_{i=2}^{N} \mathbf{A_i \sin(\omega_i t)}$ **contains** $\mathbf{\omega_i = \omega_1}$**.** Without loss of generality, we assumed that $\omega_i = \omega_1$ for $i = 2$.

The sum of the two equi-frequent sine functions could be expressed as:

$$\mathbf{(3.3.11) A_1 \sin(\omega_1 t) + A_2 \sin(\omega_1 t + \varphi) = \sqrt{A_1^2 + A_2^2 + 2A_1 A_2 \cos(\varphi)} \cdot \sin(\omega_1 t + \vartheta)}$$

Where φ is the relative delay between the two functions and ϑ is given by:

$$\vartheta = \tan^{-1}\left(\frac{A_2 \sin(\varphi)}{A_1 + A_2 \cos(\varphi)}\right) + \begin{cases} 0 & A_1 + A_2\cos(\varphi) \geq 0 \\ \pi & A_1 + A_2\cos(\varphi) \leq 0 \end{cases}$$

And the RMS would be given by:

$$(3.3.12) X_3 = \sqrt{\frac{\delta}{(K-\omega_1^2 M)^2+\omega_1^2 B^2}+\sum_{i=3}^{N}\frac{A_i^2}{(K-\omega_i^2 M)^2+\omega_i^2 B^2}}$$

Where $\delta = A_1^2 + A_2^2 + 2A_1A_2\cos(\varphi)$

The derivation of sensitivity with respect to ω_1 for this expression is very similar to the one we performed in Section 3.2.

The final expression for this case is given by:

$$\boldsymbol{(3.3.13) S_3 = \frac{S_{1_{\omega_1}}}{1+\underbrace{\sum_{i=3}^{N}\frac{A_i^2\alpha}{\delta\alpha_i}}_{>0}} < S_{1_{\omega_1}}}$$

The redundant version of this case (testing sensor contains a single sine function) is quite similar to the original case expressed by Equation (3.3.2), with the only difference being the amplitude of the sine power source:

$X_4 = \frac{\sqrt{\delta}}{\sqrt{\left(K-\omega_1^2 M\right)^2+\omega_1^2 B^2}}$ Where $\delta = A_1^2 + A_2^2 + 2A_1A_2\cos(\varphi)$

Hence, the sensitivity of the system to frequency change with addition of an equi-frequent sine power source remains the same.

3.3 Sensitivity to Amplitude

The original expression for a **single sine power source** is:

$$(3.4.1)\ X_1 = \frac{1}{\sqrt{2}}\frac{A}{\sqrt{(K-\omega_1^2 M)^2+\omega_1^2 B^2}}$$

System sensitivity to amplitude is given by:

$$(\mathbf{3.4.2})\boldsymbol{S_{1\_A}} = \left|\frac{\boldsymbol{\partial X_1}}{\boldsymbol{\partial A}}\cdot\frac{\boldsymbol{A}}{\boldsymbol{X_1}}\right| = \left|\frac{\boldsymbol{A}}{\sqrt{\boldsymbol{\alpha}}}\cdot\frac{\sqrt{\boldsymbol{\alpha}}}{\boldsymbol{A}}\right| = \mathbf{1}$$

The sensitivity for a testing sensor consisting of springs, dampers, and a weighted sum of single sine power source ($\omega_i \neq \omega_1 \ \forall \ i > 1$) is:

$$X_4 = \frac{1}{\sqrt{2}}\sqrt{\frac{A_1^2}{\alpha_1} + \sum_{i=2}^{N}\frac{A_i^2}{\alpha_i}}$$

$$(\mathbf{3.4.3})\ S_4 = \left|\frac{\partial X_4}{\partial A_1}\cdot\frac{A_1}{X_4}\right| = \boldsymbol{S_4} = \frac{\boldsymbol{S_{1\_A}}}{\mathbf{1} + \underbrace{\sum_{i=2}^{N}\frac{A_i^2\alpha_1}{A_1^2\alpha_i}}_{>0}} < \boldsymbol{S_{1\_A}}$$

The redundant version of this case [single sine power source – Equation (3.3.2)] was calculated from this result by taking $A_i = 0 \ \forall i > 2$ and denoting $B = A_2, \beta = \alpha_2$:

$$(3.4.5)\ \ S_5 = \frac{S_{1\_A}}{1 + \underbrace{\frac{B^2\alpha}{A^2\beta}}_{>0}} < S_{1\_A}$$

Now, if $\sum_{i=2}^{N} \boldsymbol{A_i sin(\omega_i t)}$ **contains**$\boldsymbol{\omega_i = \omega_1}$:

$$X_3 = \frac{1}{\sqrt{2}}\sqrt{\frac{\delta}{\alpha_1} + \sum_{i=3}^{N}\frac{A_i^2}{\alpha_i}}$$

Deriving with respect to $\mathrm{A_1}$ and multiplying by $\mathrm{A_1/X_3}$ will lead to:

$$(3.4.6)\mathrm{S_6} = \left|\frac{\partial \mathrm{X_3}}{\partial \mathrm{A_1}}\cdot\frac{\mathrm{A_1}}{\mathrm{X_3}}\right| = \frac{\mathrm{S_{1_A}}}{\left|1 + \frac{\mathrm{A_2^2 + A_1A_2\cos(\varphi)}}{\mathrm{A_1^2 + A_1A_2\cos(\varphi)}} + \frac{\gamma}{(\mathrm{A_1^2 + A_1A_2\cos(\varphi)})}\right|}$$

Where $\gamma = \sum_{\mathrm{i}=3}^{\mathrm{N}}\frac{\mathrm{A_i^2\alpha_1}}{\alpha_\mathrm{i}}$ and $\mathrm{S_{1_A}} = 1$.

To prove our claim, we needed to analyze the denominator and verify that it was indeed greater than one. For this purpose, we analyzed the opposite scenario and showed that the condition for this scenario could not be fulfilled:

$$\left|1+\frac{A_2^2+A_1A_2\cos(\varphi)}{A_1^2+A_1A_2\cos(\varphi)}+\frac{\gamma}{(A_1^2+A_1A_2\cos(\varphi))}\right|<1$$

$$\text{Hence:}\frac{1}{A_1^2+A_1A_2\cos(\varphi)}(A_2^2+A_1A_2\cos(\varphi)+\gamma)<0$$

For this to be possible, we need our equation to fulfill the following conditions:

Either $A_1^2+A_1A_2\cos(\varphi)<0$ and $(A_2^2+A_1A_2\cos(\varphi)+\gamma)>0$ or vice versa.

Now, let us analyze each part individually:

$A_1^2+A_1A_2\cos(\varphi)<0 \leftrightarrow \cos(\varphi)<-\frac{A_1}{A_2}andA_2>A_1$because of the cosine function.

Thus, we have:(a) $\cos(\varphi)<-\frac{A_1}{A_2}$ and (b)$A_2>A_1$

Now, the second expression:

$$(A_2^2+A_1A_2\cos(\varphi)+\gamma)>0 \leftrightarrow \cos(\varphi)>-\frac{\gamma+A_2^2}{A_1A_2}$$

For this to be possible, we need $A_1A_2>\gamma+A_2^2 \leftrightarrow A_1>\frac{\gamma}{A_2}+A_2$

But $\gamma=\sum_{i=3}^{N}\frac{A_i^2\alpha_1}{\alpha_i}$ where $\alpha_i=(K-\omega_i^2M)^2+\omega_i^2B^2>0$,

so $\gamma>0$ and inevitably $A_1>A_2$.

This is a contradiction to (b).

In the borderline case where:

$$\frac{1}{A_1^2+A_1A_2\cos(\varphi)}(A_2^2+A_1A_2\cos(\varphi)+\gamma)=0$$

We need:

$\cos(\varphi)=-\frac{A_1}{A_2}and\cos(\varphi)=-\frac{\gamma+A_2^2}{A_1A_2}$

Once again, we have a contradiction. From the first expression, we receive $A_2 \geq A_1$, but from the second:

$$A_1 \geq \frac{\gamma}{A_2} + A_2 > A_2.$$

Hence, **the sensitivity is reduced.**

For the redundant case where $\gamma = 0$, the sensitivity will be reduced in all cases **but $\cos(\varphi) = -1$ and $A_1 = A_2$, for which it will remain the same.**

In addition to the proof of the principle for the frequency and amplitude of the tested sine power source, we analyzed an expansion for additional system parameters, and we determined the conditions for enhanced sensitivity to damper and spring constants. Those conditions are presented in Appendix B.

4. Demonstration of the concept

4.1 Simulation of dynamical system interaction in a noisy environment

To demonstrate the active sensing principle, we constructed a classification task based on data from a computer simulation, closely related to the system depicted in Section 3. We employed two testing models closely resembling the models depicted in Section 3 (active and passive scenarios). The tested parametric families were based on each of the two models, where every "family member" corresponds to a different combination of parameter values in the system:

a. The "passive family": A linear spring-dashpot (Karniel and Inbar 2000) – $Force = Kx + B\dot{x}$. The parameters K and B are assigned one of five predetermined pairs of values ($K \in \{50,40,30,20,10\}, B \in \{0,1,2,3,4\}$).

b. The "active family": A sine with linear spring-dashpot – $Force = \sin(2\pi f t) + Kx + B\dot{x}$. Values of K and B are as in the linear spring-dashpot model. Sine frequency is $f \in [1.5, 2.5]$. The frequency range was selected to correlate with previous data analysis which was done on recordings of subjects' handshake movements administered through a Phantom® Desktop haptic device™ (SensAble Technologies Inc) (Karniel, Avraham et al. 2010; Avraham, Nisky et al. 2012).

During the simulation, each of the models performed interactions with all members of the tested parametric family (mitigated by a 1 kg mass). Features based on the mass trajectory such as maximal force and velocity, position change frequency, mean acceleration, jerk, and energy were extracted from each simulation and utilized to discriminate between pairs of different members of the same parametric family (data from two different interactions with the same testing model used for a binary classification task). The score for each model was given based on the total classification performance of all possible combinations of the tested parametric family.

4.2 Testing the classification performance of the models

Following the simulation, we performed a classification process assessing each model type for its efficiency in discriminating between family members of the same parametric family (graded separately for discrimination of each of the five families). For this purpose, we used a linear classifier (Bishop 2006). This type of classification is based on the assumption that the output result of the classifier (target) can be expressed as a weighted sum of the inputs:

$$(4.2.1)\mathrm{Y_j} = \sum_{\mathrm{i=1}}^{\mathrm{N}} \mathrm{W_{ij}} \cdot \mathrm{U_i} + \mathrm{W_0}$$

[can be also written as $Y = W^T U$, where U is the augmented input (feature) vector – $\begin{bmatrix} \overline{u_1} \\ \overline{u_2} \\ \dots \\ \overline{u_n} \\ \overline{1} \end{bmatrix}$]

W_0 serves as intentional biases.

Each weight in this model represents the relation between a specific input and output pair (Karniel and Inbar 1999). We chose a Mean Square Error (MSE) cost function and utilized steepest descent (gradient descent) to determine the weights. All simulations and data analyses were performed utilizing MATLAB 2009 b.

4.3 Classification Results

The classification results (Fig. 4.1) clearly demonstrate our principle; one can see that the sine-spring-damper active sensor performed significantly better than the spring-damper sensor in the spring-damper parametric family (passive-tested system) and that the passive sensor performed significantly better than the active sensor in the sine-spring-damper parametric family (active-tested system).

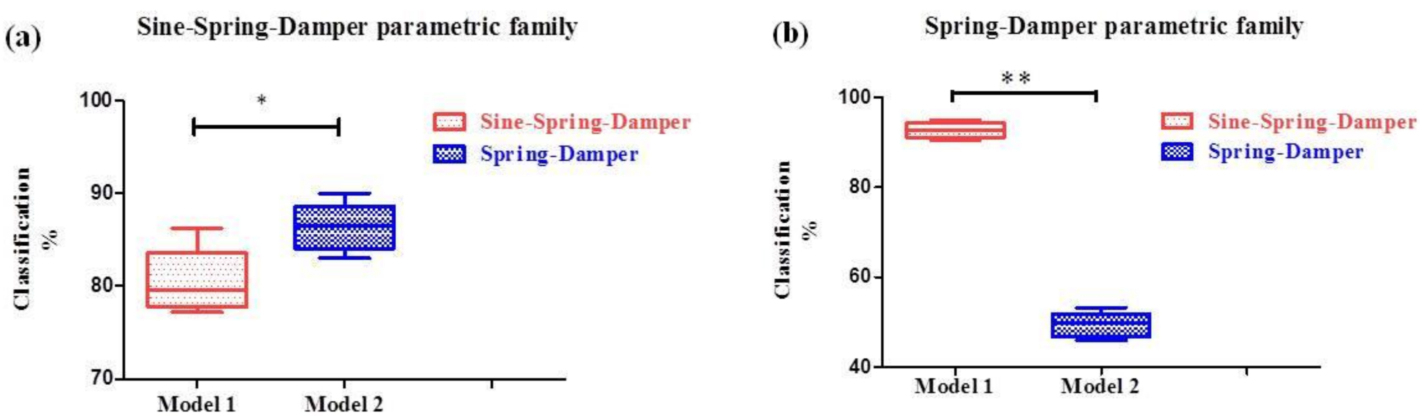

Figure 4.3.1 Simulation results: Classification performance of sine-spring-damper and spring-damper sensors interacting vs. spring-damper (**Right**) and sine-spring-damper (**Left**) parametric families.

5. Manual discrimination experiment

5.1 General

In this section, we consider a possible extention of the proposed principle beyond the linear case by examining human–robot interactions. In this interaction,the simulated object is linear, but the interogator (human) is probably not. To accomplish this goal, we carried out a slightly simplified variant of the classical manual discrimination experiment (Berliner and Durlach 1973; Pang, Tan et al. 1991; Tan, Durlach et al. 1995), performing a force stimuli discrimination task utilizing a robotic haptic device.

5.2 Subjects and apparatus

Ten subjects, four males and six females (S1-S10, all students at the Ben-Gurion University of the Negev, 24–35 years old, average age of 27) were paid to participate in the experiment. All participants were right-handed and without any known motor impairment. Each subject signed an informed consent form to participate in the experiment, as stipulated by the institutional Helsinki committee, Beer-Sheva, Israel.

During the experiment, subjects held the handle of a robotic arm (SensAble PHANTOM® Desktop™ Haptic Device), which produces forces according to preprogrammed software and samples the time and position of the handle base at 100 Hz. Each subject was seated on a chair facing the robotic arm with his or her right arm placed on the right armrest, with the elbow in the middle so that the front part of the forearm extended over the armrest in the direction of the robotic device. A 21" LCD computer screen was placed at approximately 45° to the left of the chair at eye level. The subject's left palm was placed on the control keyboard (see Fig 5.3.1). Subjects were equipped with ear plugs to eliminate any possible auditory cues arising from the motion of the robotic arm and were instructed to focus on the computer screen during each interaction, reducing possible visual cues.

5.3 Stimuli

While the subjects held the handle of the haptic device during the trial, they experienced forces that were a combination of a spring, a damper, and a sine power source set in a configuration similar to that described for the tested system in Section 3.2.

The values of the different parameters of the experiment are presented in Table 1. The values of frequency and amplitude for the power source $A\sin(\omega_1 t)$, as well as spring and damper constants were chosen based on previously performed research on the characteristics of the human handshake (Karniel, Avraham et al. 2010; Karniel, Nisky et al. 2010).

During each interaction, the haptic device presented the subject with one of two possible stimuli defined by the amplitude of the sinusoidal power source: ***reference*** amplitude A or ***examined*** amplitude $A + \Delta A$, where ΔA is a positive increment.

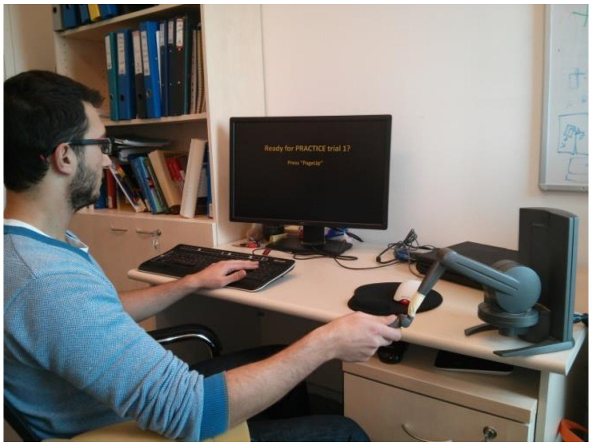

Figure 5.3.1: The experimental setup. Subject sat on a chair facing the SensAble PHANTOM® Desktop™ Haptic Device. During the experiment, the subject was instructed to look at the computer screen in order to reduce possible visual cues.

5.4 Procedure

A one interval two-alternative forced-choice paradigm was applied (Macmillan and Creelman 2004). For each trial, the power source of the tested system had either reference amplitude A or examined amplitude of $A + \Delta A$, both given with equal a priori probability. Subjects were informed that they were going to experience one of two possible systems, either a reference system applying low force or a test system applying higher forces. A five-second trial was followed by the subject’s answer to the question, "Was the force high, or was it low?" Instructions on the computer screen directed subjects to push the “home” button for low force and “end” for high force. Following a subject’s choice, correct answer feedback was given.

An experiment run consisted of 16 practice trials followed by 64 test trials, and took 15–20 minutes. The increment on each run was set to one of four possible values: 10, 15, 20, or 25 percent and kept constant throughout the run. The complete experimental session was composed of four runs (one for each value), with the increment changes arranged in an ascending order. One experimental session took up to 1.5 hours.

Two exploratory conditions were tested, referred to as “active” and “passive.” Every participant performed the complete experimental session twice on adjacent days, once for each condition. During the “passive” condition, subjects were instructed to be led by the robot without applying any additional force. During the “active” condition, the subjects were instructed not to remain passive, i.e., to test the forces produced by the haptic device while applying force on the handle of the robotic arm. The subjects were allowed to choose any self-applied force to assist themselves to perform the task at hand, and the order of the sessions was randomized between subjects. Half of the subjects began with the “active” condition session the first day, followed by the “passive” session the second, and the other half vice versa.

A total of 320 trials was collected for each participant during a single experimental session, while a total of 6,400 trials was collected during the entire experiment. Parameter values used can be found in Table 1.

Table I: Parameter values for the experiment

Parameter	A_0(N)	$\Delta A/A_0$ (%)	K [N/M]	B[N·m/Sec]	ω_1[Rad/Sec]
Value	1	10,15,20,25	20	10	10

5.5 Data analysis

Subjects' answers for each experimental condition were used to calculate the estimates of sensitivity index d', response bias β, and the jnd from the stimulus–response matrix (Berliner and Durlach 1973). In this analysis method, the underlying density functions represent the sensory process related to a two-stimuli discrimination task assumed to be normal with means M_1 and M_2 and of equal variance σ.

The sensitivity index d', defined as the normalized distance between the two means, can be formulated as follows:

$$(5.5.1)\ d' = (M_1 - M_2)/$$

And the response bias:

$$(5.5.2)\ \beta = [c - (M_1 - M_2)]/2\sigma$$

representing the normalized distance between the response criterion c and the midpoint between the expected values of the distributions.

Given that the alteration ind' value is approximately proportional to the increment $\Delta A/A$, the slope $\delta = d'/(\Delta A/A)$ can serve as an estimate of the subjects' performance level. Furthermore, if the results are characterized by an unbiased response behavior ($\beta = 0$), a 75% correct performance threshold would conform to $d' = 1$, providing a straightforward Weber fraction (jnd %) estimate as the inverse of the slope δ at that point.

The WF for each participant was estimated as the inverse of the slope $\delta(d' = 1)$, averaged over all increments (for the tested reference force amplitude A) for each participant. The mean WF for each experimental condition was derived from the individual WFs pooled across all subjects' results for that condition. See Tan et al. (1995) for more details.

Another procedure, utilized by Israr et al. (2009), provides a slight alteration of the above-mentioned method. According to this procedure, participants' WF is estimated by first calculating the inverse of $\delta(d' = 1)$(WF) for each increment value and then averaging overall WF values, interchanging the order of actions of the previous method. In our case, any significant result with respect to the jnd was validated for both mentioned procedures.

Subject performance was tested on four increments: 10, 15, 20, and 25%. Because all subjects showed significantly different behavior in the last increment during the active condition (possibly due to fatigue), only the first three increments were subsequently utilized to estimate the jnd.

To test the significance of jnd differences between the "active" and "passive" conditions, we utilized a paired one-tailed bootstrap analysis (10,000 repetitions), as well as a Wilcoxon signed-rank test. The choice of nonparametric methods arose not only due to the relatively small sample size, but also because each group included a mixture of jnd scores from both the first and second days of the experiment (primary or repeated interactions with the robot).

5.6 Results

All ten subjects performed better in the passive condition, as predicted by the proposed active sensing principle (Fig. 5.1). A summary of the results including jnd and response bias values for each condition is given in Table 2.

The response bias was found to be negligible both overall and for each of the experimental conditions separately, with the total response bias average of $\beta_{Tot} = -0.0019$ and bootstrap 95% interval of [-0.0612 0.0620], response bias for the passive condition of $\beta_{Passive} = -0.0508$ and bootstrap 95% interval of [-0.1403 0.0461], and for the active condition of $\beta_{active} = -0.0407$ and bootstrap 95% interval of [-0.0214 0.1431]. It seems the subjects had a slight (but not significant) tendency to more often answer "low force" when tested in the passive condition and vice versa for the active condition.

Table II: Summary of Results						
	JND (%)				Response Bias	
	Method 1		*Method 2*			
Condition	**Average**	**Std. Error**	**Average**	**Std. Error**	**Average**	**Std. Error**
All	18	2.03	22.85	2.05	-0.0019	0.0313
Active	22.46	3.15	28.09	6.54	0.047	0.0466
Passive	13.54[a]	1.63	15.12[b]	2.18	-0.0508	0.0402

[a] Significantly different from mean 'active' value (Method 1, p=0.005). [b] significantly different from mean 'active' value (Method 2, p<0.005)

Table 2: Summary of results. Mean and standard error are given for response bias and jnd%. Results concerning jnd are given for the two methods of calculation mentioned above. Method 1— following Tan et al. (1995), Method 2— following Israr et al (2009).

The mean jnd % value for the passive condition (average of 13.54% with std. of 5.5%) was found to be significantly lower [10,000 repetitions one-tailed bootstrap (Efron and Efron 1982), *p*<0.004, and one-tailed Wilcoxon signed-rank test for the median (Wilcoxon and Wilcox 1964), with *p*=0.005] than that of the active condition (average of 22.46 % with std. of 8.44%). The results are presented in Figure 5.6.1.

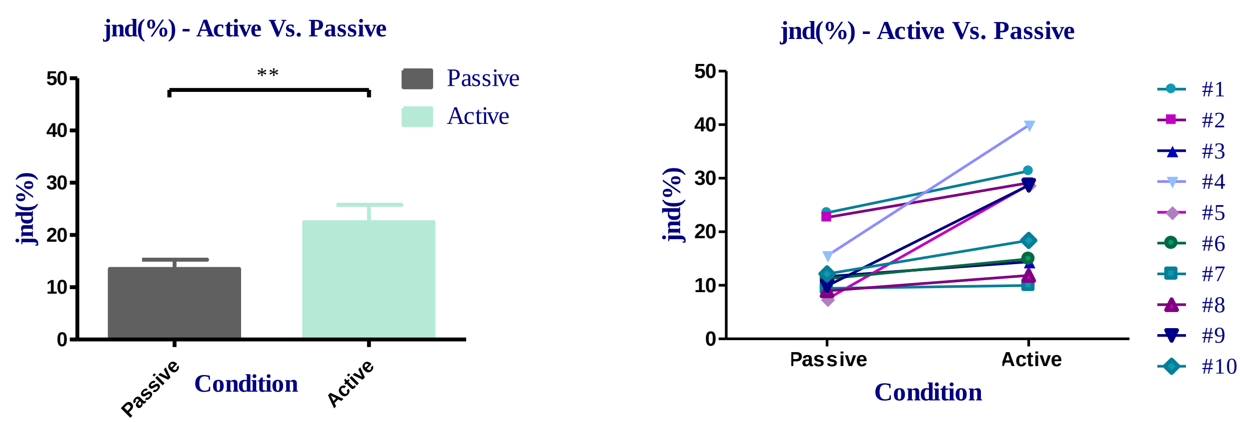

Figure 5.6.1: Left: jnd% for passive and active conditions estimated for each of the ten subjects. All subjects performed better in the passive condition. **Right:** Mean jnd% and standard error on each of the conditions. The mean jnd% value for the passive condition is significantly lower than that estimated in the active condition (10,000 repetitions one-tailed bootstrap, *p*<0.004).

6. Discussion

In this paper, we have presented the active sensing principle for haptic interaction with dynamical systems: to act on a passive system and to absorb information from an active system in order to maximize sensitivity and, therefore, classification performance. We have provided an analytical proof of the principle for a linear second-order mechanical system. Additionally a computer simulation example demonstrated the principle in a noisy environment. Finally, we performed a psychophysical experiment comparing subjects' discrimination performance in active and passive conditions. Our results support the proposed principle.

In Section 2, we formulated the problem of active sensing, articulating the merits of improved sensitivity in the context of a classification problem. Improved sensitivity is without doubt an eminent quality and plausible cost measure for interaction with dynamical systems. Notwithstanding, it is important to note that the specific choice of a cost measure is directly related to the nature of the task at hand. Hence, the expansion of the proposed principle to additional active sensing scenarios may require the consideration of adjustment or alteration of the discussed cost measure.

This study employed phasor analysis, typically used to study the steady state. However, since our results are valid for the sum of sinusoidal functions, one can use Fourier analysis and conclude that our results are also valid for a variety of force functions. Nevertheless, in the analysis of this study, we did not address the possibility to incorporate feedback or to study the system by perturbing it and inspecting how it returns to equilibrium.

The diminishing effect of feedback on system sensitivity to plant parameter variation (Mazer 1960; Cruz Jr. and Perkins 1964; Kreindler 1968) and its effect on impedance (Blackman 1943; Rosenstark 1974) has been demonstrated in the literature. However, in active sensing, one can control the feedback and, therefore, it is possible that a specific active sensing strategy may enhance sensitivity. The study of system characteristics through the introduction of perturbation is a very common procedure of control theory. The assumptions we have made in our proof in Section 3 (i.e., steady state and power source represented as the weighted sum of sinusoidal functions) impose some restrictions on the incorporation of perturbation analysis in the current scope. Nonetheless it is important to note that some forms of perturbations could be addressed since they can be approximated to a given precision as a sum of sinusoidal functions.

In Section 3 and Appendix B, we analyzed the relative sensitivity of active and passive probing sensors for various parameters of the tested system. In the case of frequency and amplitude of the tested power source, our analysis leads to an explicit conclusion. An inspection of the calculations for the case of spring and damper parameters shows that sensitivity to these parameters of the active-tested system can be improved by introduction of an active power source to the probing sensor, granted that specific conditions with respect to the relation and range of testing and tested system parameter values are fulfilled. Note, that although improved sensitivity in an active–active scenario is theoretically possible, it is constricted to a specific range of values. Beyond this range, the active power source addition will lead, as in the previous case, to diminished sensitivity. Furthermore, fulfillment of the specified conditions requires the adjustment of our testing system parameters in accordance of those of the tested system. Therefore, one can extend our principle beyond the cases proven in the body of this paper to other more general conditions such as the simulated noisy condition and a wide range of other systems by replacing the deterministic notations with stochastic ones, asserting that the expectation of the sensitivity should increase by following the active sensing principle.

In Section 4, we demonstrated the proposed principle via computer simulation. The parameter range was chosen to reflect the range of human movement in the simulated task, as it was empirically found in a previous study related to a Turing-like test for motor intelligence (Karniel, Avraham et al. 2010; Karniel, Nisky et al. 2010).

The conditions presented in the noisy simulation example as well as in the psychophysical experiment of human robot interaction were not analytically validated. The results supporting the principle beyond the linear case encourage future studies to extend its rigorous proof to cases of interaction with higher order nonlinear systems, in general, and with haptic devices, in particular.

It is important to note that in order to accurately estimate the jnd, a considerably longer experiment design with the WF calculated at different reference forces is required. Notably, a large number of experimental run repetitions would be desired to account for the possible effect of training. Moreover, this type of design would enable the felicitous validation of additional conditions. Among these conditions are an affirmation that the rectified receiver operation curves are straight lines with a united slope and that the temporal variability of d' values is consistent with the Bernoulli process assumption. A good example for such a meticulous validation process can be found in Pang, Tan et al. (1991).

In the framework of our simplified experimental procedure, we did not account for the conditions mentioned in the above mentioned study as we focused only on the fulfillment of cardinal elements such as the proportionality between $d'$and the increments and a negligible response bias. Nevertheless, it is important to bear in mind that our primary goal was not the estimation of the jnd for a novel environment or property (force/compliance, etc.) but to test our active sensing hypothesis by comparing the jnd of active and passive human discrimination.

Performance of additional experiments, following the guidelines mentioned above, would allow a more precise estimation of the jnd for the described discrimination task. Notwithstanding, disregarding the accuracy of the jnd estimation, the methodology used here is sufficient for a valid comparison of performance between the active and passive conditions.

The proposed principle could be useful for various applications in the field of active sensing and robot–human interaction, in general, and in the study of human interaction with external dynamic systems, in particular, providing us with additional insight to human perception of dynamical systems. Whether humans actually follow this principle, is still an open question of great interest to the motor neuroscience community.

Acknowledgment

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Figures

Fig. 2.1: Output of a system Y as a function of an inner parameter X (contaminated with white Gaussian noise at 1 dB). Blue and red solid lines represent two possible relations between system output and one of its parameters. Dotted and dashed lines represent the changes in the value of output for each of the two relations for similar values of parameter X.

Fig. 3.1: Second-order mechanical system consisting of a spring, a damper, a sine power source, and a mass interacting with a spring, a damper, and an optional power source.

Fig. 3.2: Claim: Introduction of an additional power source to a testing system will result in a reduction of the sensitivity of the system.

Fig. 4.3.1: Simulation results: Classification performance of Sine-Spring-Damper and Spring-Damper sensors interacting vs. Spring-Damper (**Right**) and Sine-Spring-Damper (**Left**) parametric families.

Fig. 5.3.1: **The experimental setup**. Subject sat on a chair facing the SenseAble PHANTOM® Desktop™ Haptic Device. During the experiment, the subject was instructed to look at the computer screen in order to reduce possible visual cues.

Figure 5.6.1: Left: jnd% for passive and active conditions estimated for each of the ten subjects. All subjects performed better in the passive condition. **Right:** Mean jnd% and standard error for each of the conditions. The mean jnd% value for the passive condition is significantly lower than that estimated for the active condition (10,000 repetitions one-tailed bootstrap, $p<0.004$).

Tables

Table I: Parameter values for the experiment					
Parameter	A_0(N)	$\Delta A/A_0$ (%)	K[N/M]	B[N·m/Sec]	ω_1[Rad/Sec]
Value	1	10,15,20,25	20	10	10

Table II: Summary of Results

	jnd (%)				Response Bias	
	Method 1		*Method 2*			
Condition	**Average**	**Std. Error**	**Average**	**Std. Error**	**Average**	**Std. Error**
All	18	2.03	22.85	2.05	-0.0019	0.0313
Active	22.46	3.15	28.09	6.54	0.047	0.0466
Passive	13.54[a]	1.63	15.12[b]	2.18	-0.0508	0.0402

[a] Significantly different from mean "active" value (Method 1, p=0.005). [b] Significantly different from mean "active" value (Method 2, p<0.005)

Table 2: Summary of results. Mean and standard error are given for response bias and jnd%. Results concerning jnd are given for the two methods of calculation mentioned above. Method 1— following Tan et al. (1995), Method 2— following Israr et al.(2009).

Appendix A. Sensitivity-detailed calculation

This appendix provides the detailed analytical derivation for the proof of the claim for sensitivity to frequency and amplitude (Section 3.3).The differential equation of the mass trajectory (no power source in testing system) is given by:

$$(\mathbf{6.1})\boldsymbol{M\ddot{X}_1 + B\dot{X}_1 + KX_1 = Asin(\omega_1 t)}$$

Where X_1 is the mass trajectory, $B = (B_1 + B_2)$, $K = K_1 + K_2$, and $\omega_1 = 2\pi f_1$.
The RMS of the output is given by:

$$(\mathbf{6.2})\boldsymbol{X_1} = \frac{\boldsymbol{A}}{\sqrt{\mathbf{2}\left[\left(\boldsymbol{K - \omega_1^2 M}\right)^{\mathbf{2}} + \boldsymbol{\omega_1^2 B^2}\right]}}$$

We denote

$$\alpha_1 = (K - \omega_1^2 M)^2 + \omega_1^2 B^2$$

And derive the systems sensitivity:

$$S_{1_{\omega_1}} = \left|\frac{\partial X_1}{\partial \omega_1} \cdot \frac{\omega_1}{X_1}\right| =$$

$$= \left| -\frac{A \cdot (-4\omega_1 M(K - \omega_1^2 M) + 2\omega_1 B^2)}{2 \cdot \alpha^{\frac{3}{2}}} \cdot \frac{\omega_1 \cdot \alpha^{\frac{1}{2}}}{A} \right|$$

$$(6.3) \boldsymbol{S_{1\_\omega_1}} = \left| \frac{\boldsymbol{(2\omega_1 M\left(K - \omega_1^2 M\right) - \omega_1 B^2)}}{\boldsymbol{\alpha}} \cdot \boldsymbol{\omega_1} \right|$$

Now, let us consider an active probing sensor with $\mathrm{f} = \mathrm{B}\sin(\omega_2 \mathrm{t} + \varphi)$.

The updated differential equation:

$$(6.4) \mathbf{M\ddot{X}_2 + B\dot{X}_2 + KX_2 = A\sin(\boldsymbol{\omega_1} t) + B\sin(\boldsymbol{\omega_2} t + \boldsymbol{\varphi})}$$

$$(\omega_1 \neq \omega_2)$$

And the RMS of the output is:

$$(6.5) \mathbf{X_2} = \frac{\mathbf{1}}{\sqrt{\mathbf{2}}} \sqrt{\frac{\mathbf{A^2}}{\left(\mathbf{K} - \boldsymbol{\omega_1^2} \mathbf{M}\right)^2 + \boldsymbol{\omega_1^2} \mathbf{B^2}} + \frac{\mathbf{B^2}}{\left(\mathbf{K} - \boldsymbol{\omega_2^2} \mathbf{M}\right)^2 + \boldsymbol{\omega_2^2} \mathbf{B^2}}}$$

We denote

$$\beta = (\mathrm{K} - \omega_2^2 \mathrm{M})^2 + \omega_2^2 \mathrm{B}^2$$

The updated sensitivity to a change in frequency would be:

$$S_2 = \left| \frac{\partial X_2}{\partial \omega_1} \cdot \frac{\omega_1}{X_2} \right|$$

$$= \left| \frac{A^2 \cdot \frac{(2\omega_1 M(K - \omega_1^2 M) - \omega_1 B^2)}{\alpha^2}}{\left(\frac{A^2}{\alpha} + \frac{B^2}{\beta}\right)} \cdot \omega_1 \right|$$

$$= \left| \frac{A^2 \cdot (2\omega_1 M(K - \omega_1^2 M) - \omega_1 B^2)}{\alpha \cdot \alpha \left(\frac{A^2}{\alpha} + \frac{B^2}{\beta}\right)} \cdot \omega_1 \right|$$

Presenting the sensitivity S_2 in terms of $S_{1\omega_1}$:

$$(\mathbf{6.6})\boldsymbol{S_2} = \left| \frac{\boldsymbol{A^2 \cdot S_{1_{\omega_1}}}}{\boldsymbol{\alpha} \cdot \left(\frac{\boldsymbol{A^2}}{\boldsymbol{\alpha}} + \frac{\boldsymbol{B^2}}{\boldsymbol{\beta}}\right)} \right| = \frac{\boldsymbol{S_{1_{\omega_1}}}}{\mathbf{1} + \underbrace{\frac{\boldsymbol{B^2}}{\boldsymbol{A^2}} \cdot \frac{\boldsymbol{\alpha}}{\boldsymbol{\beta}}}_{\geq \mathbf{0}}} \leq \boldsymbol{S_{1\omega_1}} \xrightarrow{\boldsymbol{\omega_1 \neq 0}} (\mathbf{7})\boldsymbol{S_2} = \frac{\boldsymbol{S_{1\_\omega_1}}}{\mathbf{1} + \underbrace{\frac{\boldsymbol{B^2}}{\boldsymbol{A^2}} \cdot \frac{\boldsymbol{\alpha}}{\boldsymbol{\beta}}}_{>0}} < \boldsymbol{S_{1\omega_1}}$$

Following the addition of a weighted sum of a single sine power source,

the updated differential equation is:

$$(\mathbf{6.7})\mathbf{M\ddot{X}_1 + B\dot{X}_1 + KX_1 = A\sin(\omega_1 t) + \sum_{i=2}^{N} A_i \sin(\omega_i t)}$$

The RMS is given by:

$$(\mathbf{6.8})\mathbf{X_2} = \frac{\mathbf{1}}{\sqrt{\mathbf{2}}} \sqrt{\frac{\mathbf{A_1^2}}{\left(\mathbf{K - \omega_1^2 M}\right)^2 + \mathbf{\omega_1^2 B^2}} + \frac{\mathbf{A_2^2}}{\left(\mathbf{K - \omega_2^2 M}\right)^2 + \mathbf{\omega_2^2 B^2}} + \ \ldots}$$

$$= \frac{\mathbf{1}}{\sqrt{\mathbf{2}}} \sqrt{\sum_{\mathbf{i=1}}^{\mathbf{N}} \frac{\mathbf{A_i^2}}{\left(\mathbf{K - \omega_i^2 M}\right)^2 + \mathbf{\omega_i^2 B^2}}}$$

The sensitivity:

$$\mathrm{S_2} = \left| \frac{\partial \mathrm{X_2}}{\partial \omega_1} \cdot \frac{\omega_1}{\mathrm{X_2}} \right| =$$

$$\left| -\frac{\mathrm{A_1}^2 \cdot \frac{\left(4\omega_1 \mathrm{M}\left(\mathrm{K} - \omega_1^2 \mathrm{M}\right) - 2\omega_1 \mathrm{B}^2\right)}{\left(\left(\mathrm{K} - \omega_1^2 \mathrm{M}\right)^2 + \omega_1^2 \mathrm{B}^2\right)^2}}{2 \cdot \sum_{\mathrm{i=1}}^{\mathrm{N}} \frac{\mathrm{A_i^2}}{\left(\mathrm{K} - \omega_i^2 \mathrm{M}\right)^2 + \omega_i^2 \mathrm{B}^2}} \cdot \omega_1 \right|$$

$$= \left| \frac{{A_1}^2 \cdot \frac{\left(2\omega_1 M\left(K-\omega_1^2 M\right)-\omega_1 B^2\right)}{\alpha^2}}{\frac{A_1^2}{\alpha}\left(1+\sum_{i=2}^{N}\frac{A_i^2 \alpha}{A_1^2 \alpha_i}\right)} \cdot \omega_1 \right|$$

Where $\alpha_i = (K-\omega_i^2 M)^2 + \omega_i^2 B^2$

As before, we notice:

$$(\mathbf{6.10})\mathbf{S_2} = \frac{\mathbf{S_{1\omega_1}}}{\mathbf{1} + \underbrace{\sum_{i=2}^{N}\frac{A_i^2 \alpha}{A_1^2 \beta_i}}_{>0}} < \mathbf{S_{1\omega_1}}$$

Now, we address the case when the expression $\sum_{i=2}^{N} \mathbf{A_i \sin(\omega_i t)}$**contains** $\boldsymbol{\omega_i = \omega_1}$**.**

Without loss of generality, let us assume that $\omega_i = \omega_1$for $i = 2$.

The sum of the two equi-frequent sine functions could be expressed as:

$$(\mathbf{6.11})\mathbf{A_1 \sin(\omega_1 t)} + \mathbf{A_2 \sin(\omega_1 t + \varphi)}$$

$$= \sqrt{\mathbf{A_1^2 + A_2^2 + 2A_1A_2\cos(\varphi)}} \cdot \mathbf{\sin(\omega_1 t + \vartheta)}$$

Where φ is the relative delay between the two functions and ϑ is given by:

$$\vartheta = \tan^{-1}\left(\frac{A_2 \sin(\varphi)}{A_1 + A_2\cos(\varphi)}\right) + \begin{cases} 0 & A_1 + A_2\cos(\varphi) \geq 0 \\ \pi & A_1 + A_2\cos(\varphi) \leq 0 \end{cases}$$

And the RMS would be given by:

$$(6.12)X_3 = \sqrt{\frac{\delta}{(K-\omega_1^2 M)^2 + \omega_1^2 B^2} + \frac{A_3^2}{(K-\omega_2^2 M)^2 + \omega_2^2 B^2} + \ \ldots}$$

$$= \sqrt{\frac{\delta}{(K-\omega_1^2 M)^2 + \omega_1^2 B^2} + \sum_{i=3}^{N}\frac{A_i^2}{(K-\omega_i^2 M)^2 + \omega_i^2 B^2}}$$

Where $\delta = A_1^2 + A_2^2 + 2A_1A_2\cos(\varphi)$.

The derivation of sensitivity with respect to ω_1 for this expression is very similar to the one we performed in Section 3.2.

The final expression for this case would be given by:

$$(\mathbf{6.13})\mathbf{S_3} = \frac{\mathbf{S_{1_{\omega_1}}}}{\mathbf{1} + \underbrace{\sum_{\mathbf{i=3}}^{\mathbf{N}} \frac{\mathbf{A_i^2 \alpha}}{\boldsymbol{\delta\alpha_i}}}_{>0}} < \mathbf{S_{1_{\omega_1}}}$$

The redundant version of this case, expressed by Equation (4), (testing sensor contains a single sine function), is quite similar to the original case expressed by Equation (2), with the only difference being the amplitude of the sine power source:

$$\mathrm{X}_1 = \frac{\mathrm{A}}{\sqrt{(\mathrm{K} - \omega_1^2\mathrm{M})^2 + \omega_1^2\mathrm{B}^2}}\ ,\ \mathrm{X}_4 = \frac{\sqrt{\delta}}{\sqrt{(\mathrm{K} - \omega_1^2\mathrm{M})^2 + \omega_1^2\mathrm{B}^2}}$$

Where$\delta = \mathrm{A}_1^2 + \mathrm{A}_2^2 + 2\mathrm{A}_1\mathrm{A}_2\cos(\varphi)$

Hence the sensitivity of the system to frequency change with the addition of an equi-frequent sine power source remains the same.

Sensitivity to Amplitude

The original expression for a **single sine power source** is:

$$X_1 = \frac{1}{\sqrt{2}} \frac{A}{\sqrt{(K - \omega_1^2 M)^2 + \omega_1^2 B^2}}$$

Systems sensitivity to amplitude is given by:

$$(\mathbf{6.14})\boldsymbol{S_{1\_A}} = \left|\frac{\boldsymbol{\partial X_1}}{\boldsymbol{\partial A}} \cdot \frac{\boldsymbol{A}}{\boldsymbol{X_1}}\right| = \left|\frac{\boldsymbol{A}}{\sqrt{\boldsymbol{\alpha}}} \cdot \frac{\sqrt{\boldsymbol{\alpha}}}{\boldsymbol{A}}\right| = \mathbf{1}$$

The sensitivity for a testing sensor consisting of springs, dampers, and a weighted sum of single sine power source ($\omega_i \neq \omega_1\ \forall\ i > 1$) is:

$$X_4 = \frac{1}{\sqrt{2}} \sqrt{\frac{A_1^2}{\alpha_1} + \sum_{i=2}^{N} \frac{A_i^2}{\alpha_i}}$$

$$S_4 = \left|\frac{\partial X_4}{\partial A_1}\cdot\frac{A_1}{X_4}\right| = \frac{\frac{2A_1}{\alpha_1}}{2\sqrt{\frac{A_1^2}{\alpha_1}+\sum_{i=2}^{N}\frac{A_i^2}{\alpha_i}}}\cdot\frac{A_1}{\sqrt{\frac{A_1^2}{\alpha_1}+\sum_{i=2}^{N}\frac{A_i^2}{\alpha_i}}} = \frac{A_1^2}{\alpha_1\left(\frac{A_1^2}{\alpha_1}+\sum_{i=2}^{N}\frac{A_i^2}{\alpha_i}\right)}$$

$$\mathbf{(6.15)}\boldsymbol{S_4} = \frac{\boldsymbol{S_{1\_A}}}{\mathbf{1}+\underbrace{\sum_{i=2}^{N}\frac{A_i^2\alpha_1}{A_1^2\alpha_i}}_{>0}} < \boldsymbol{S_{1\_A}}$$

The redundant version of this case [single sine power source - Equation (4)] is calculated from this result by taking $A_i = 0\ \forall i > 2$ and denoting $B = A_2, \beta = \alpha_2$:

$$S_5 = \frac{S_{1\_A}}{1+\underbrace{\frac{B^2\alpha}{A^2\beta}}_{>0}} < S_{1\_A}$$

Now, if $\sum_{i=2}^{N} \boldsymbol{A_i sin(\omega_i t)}$ **contains**$\boldsymbol{\omega_i = \omega_1}$:

$$X_3 = \frac{1}{\sqrt{2}}\sqrt{\frac{\delta}{\alpha_1}+\sum_{i=3}^{N}\frac{A_i^2}{\alpha_i}}$$

Deriving with respect to $\mathrm{A_1}$ and multiplying by $\mathrm{A_1/X_3}$ will lead to:

$$\mathrm{S_6} = \left|\frac{\partial \mathrm{X_3}}{\partial \mathrm{A_1}}\cdot\frac{\mathrm{A_1}}{\mathrm{X_3}}\right| =$$

$$\left|\frac{2\mathrm{A_1}+2\mathrm{A_2}\cos(\varphi)}{2\alpha_1}\cdot\frac{\mathrm{A_1}}{\frac{\mathrm{A_1^2+A_2^2+2A_1A_2}\cos(\varphi)}{\alpha_1}+\sum_{\mathrm{i=3}}^{\mathrm{N}}\frac{\mathrm{A_i^2}}{\alpha_\mathrm{i}}}\right|$$

$$= \left|\frac{1}{1+\frac{\mathrm{A_2^2+A_1A_2}\cos(\varphi)}{\mathrm{A_1^2+A_1A_2}\cos(\varphi)}+\sum_{\mathrm{i=3}}^{\mathrm{N}}\frac{\mathrm{A_i^2}\alpha_1}{\alpha_\mathrm{i}(\mathrm{A_1^2+A_1A_2}\cos(\varphi))}}\right|$$

$$= \left\{\begin{matrix}\mathrm{S_{1_A}} = 1 \\ \sum_{\mathrm{i=3}}^{\mathrm{N}}\frac{\mathrm{A_i^2}\alpha_1}{\alpha_\mathrm{i}} = \gamma\end{matrix}\right\} =$$

$$= \frac{S_{1_A}}{\left|1 + \frac{A_2^2 + A_1 A_2 \cos(\varphi)}{A_1^2 + A_1 A_2 \cos(\varphi)} + \frac{\gamma}{(A_1^2 + A_1 A_2 \cos(\varphi))}\right|}$$

To prove our claim, we need to analyze the denominator and verify that it is indeed greater than one. For this purpose, we analyzed the opposite scenario and showed that the condition for this scenario could not be fulfilled:

$$\left|1 + \frac{A_2^2 + A_1 A_2 \cos(\varphi)}{A_1^2 + A_1 A_2 \cos(\varphi)} + \frac{\gamma}{(A_1^2 + A_1 A_2 \cos(\varphi))}\right| < 1$$

$$\text{Hence: } \frac{A_2^2 + A_1 A_2 \cos(\varphi)}{A_1^2 + A_1 A_2 \cos(\varphi)} + \frac{\gamma}{(A_1^2 + A_1 A_2 \cos(\varphi))} \overset{?}{<} 0$$

$$\frac{1}{A_1^2 + A_1 A_2 \cos(\varphi)} (A_2^2 + A_1 A_2 \cos(\varphi) + \gamma) < 0$$

For this to be possible, we need our equation to fulfill the following conditions:
Either $A_1^2 + A_1 A_2 \cos(\varphi) < 0$ and $(A_2^2 + A_1 A_2 \cos(\varphi) + \gamma) > 0$ or vice versa.

Now, let us analyze each part individually:

$A_1^2 + A_1 A_2 \cos(\varphi) < 0 \leftrightarrow \cos(\varphi) < -\frac{A_1}{A_2}$ and $A_2 > A_1$ because of the cosine function.

Thus, we have: (1) $\cos(\varphi) < -\frac{A_1}{A_2}$ and (2) $A_2 > A_1$

Now, the second expression is:

$$(A_2^2 + A_1 A_2 \cos(\varphi) + \gamma) > 0 \leftrightarrow \cos(\varphi) > -\frac{\gamma + A_2^2}{A_1 A_2}$$

For this to be possible, we need $A_1 A_2 > \gamma + A_2^2 \leftrightarrow A_1 > \frac{\gamma}{A_2} + A_2$

But $\gamma = \sum_{i=3}^{N} \frac{A_i^2 \alpha_1}{\alpha_i}$ where $\alpha_i = (K - \omega_i^2 M)^2 + \omega_i^2 B^2 > 0$ so $\gamma > 0$ and inevitably $A_1 > A_2$.

This is a contradiction to (2) above.

In the borderline case where:

$$\frac{1}{A_1^2 + A_1A_2\cos(\varphi)}\left(A_2^2 + A_1A_2\cos(\varphi) + \gamma\right) = 0$$

We need:

$\cos(\varphi) = -\frac{A_1}{A_2}and\cos(\varphi) = -\frac{\gamma + A_2^2}{A_1A_2}$

Once again, we have a contradiction: From the first expression, we received $A_2 \geq A_1$, but from the second

$A_1 \geq \frac{\gamma}{A_2} + A_2 > A_2$.

Hence, **the sensitivity is reduced.**

For the redundant case where $\gamma = 0$, the sensitivity will be reduced in all cases **but $\boldsymbol{\cos(\varphi) = -1}$ and $\boldsymbol{A_1 = A_2}$, for which it will remain the same**

Appendix B. Sensitivity-additional analysis

In Section 3.3, we presented an analytical derivation for sensitivity to frequency and amplitude. In this appendix, we present an expansion of the analysis for additional system parameters and present the conditions for enhanced sensitivity to damper and spring constants.

Sensitivity to spring and damper constant

$$(7.1)X_1 = \frac{1}{\sqrt{2}}\frac{A}{\sqrt{(K_1 + K_2 - \omega_1^2 M)^2 + \omega_1^2 B^2}}$$

The system's sensitivity to K_1:

$$S_{1_{k_1}} = \left|\frac{\partial X_1}{\partial k_1} \cdot \frac{k_1}{X_1}\right|$$

$$= \left|\frac{2A(K_1 + K_2 - \omega_1^2 M)}{2[(K_1 + K_2 - \omega_1^2 M)^2 + \omega_1^2 B^2]^{\frac{3}{2}}} \cdot \frac{K_1\sqrt{(K_1 + K_2 - \omega_1^2 M)^2 + \omega_1^2 B^2}}{A}\right|$$

$$(7.2) S_{1k_1} = \left|\frac{K_1(K_1 + K_2 - \omega_1^2 M)}{\alpha}\right|$$

With the addition of a weighted sum of sine power sources ($\omega_i \neq \omega_1 \, \forall \, i > 1$):

$$X_4 = \frac{1}{\sqrt{2}} \sqrt{\frac{A_1^2}{\alpha_1} + \sum_{i=2}^{N} \frac{A_i^2}{\alpha_i}}$$

The sensitivity is:

$$S_6 = \left|\frac{\partial X_4}{\partial k_1} \cdot \frac{k_1}{X_4}\right|$$

$$= \left|\frac{\frac{2A_1^2(K_1+K_2-\omega_1^2 M)}{\alpha_1^2} + \sum_{i=2}^{N} \frac{2A_i^2(K_1+K_2-\omega_i^2 M)}{\alpha_i^2}}{2\sqrt{\frac{A_1^2}{\alpha_1} + \sum_{i=2}^{N} \frac{A_i^2}{\alpha_i}}} \cdot \frac{K_1}{\sqrt{\frac{A_1^2}{\alpha_1} + \sum_{i=2}^{N} \frac{A_i^2}{\alpha_i}}}\right| =$$

$$= \left|\frac{\frac{(K_1+K_2-\omega_1^2 M)K_1}{\alpha} + \sum_{i=2}^{N} \frac{A_i^2(K_1+K_2-\omega_i^2 M)K_1\alpha}{A_1^2\alpha_i^2}}{1 + \sum_{i=2}^{N} \frac{A_i^2\alpha_1}{\alpha_i A_1^2}}\right|$$

$$= S_{1\_k_1} \left|\frac{1 + \sum_{i=2}^{N} \frac{A_i^2(K_1+K_2-\omega_i^2 M)\alpha^2}{A_1^2\alpha_i^2(K_1+K_2-\omega_1^2 M)}}{1 + \sum_{i=2}^{N} \frac{A_i^2\alpha_1}{\alpha_i A_1^2}}\right| =$$

Once again, we explore the conditions for improved sensitivity. As we have seen, this could be rephrased in the following manner:

$$\left|\frac{1+\sum_{i=2}^{N}\frac{A_i^2(K_1+K_2-\omega_i^2M)\alpha^2}{A_1^2\alpha_i^2(K_1+K_2-\omega_1^2M)}}{1+\underbrace{\sum_{i=2}^{N}\frac{A_i^2\alpha_1}{\alpha_iA_1^2}}_{>0}}\right|\overset{?}{>}1$$

$$\sum_{i=2}^{N}\frac{A_i^2(K_1+K_2-\omega_i^2M){\alpha_1}^2}{A_1^2\alpha_i^2(K_1+K_2-\omega_1^2M)}>\sum_{i=2}^{N}\frac{A_i^2\alpha_1}{\alpha_iA_1^2}$$

$$\sum_{i=2}^{N}\frac{A_i^2}{\alpha_i}\left(\frac{(K_1+K_2-\omega_i^2M)\alpha_1}{(K_1+K_2-\omega_1^2M)\alpha_i}-1\right)>0$$

The redundant version of this case (testing the sensor power source is a single sine function) could be easily derived from this result by taking $A_i=0\ \forall i>2$ and denoting $B=A_2, \beta=\alpha_2$:

$$\underbrace{\frac{B^2}{\beta}}_{>0}\left(\frac{(K_1+K_2-\omega_2^2M)\alpha_1}{(K_1+K_2-\omega_1^2M)\beta}-1\right)>0$$

The conditions which fulfill this inequality are:

$(K_1+K_2-\omega_2^2M)\alpha_1>0$ and

$(K_1+K_2-\omega_1^2M)\beta>0$ and

$(K_1+K_2-\omega_2^2M)\alpha_1>(K_1+K_2-\omega_1^2M)\beta$

Since both α_1 and β are positive, the first two expressions could be rewritten as follows:

$$\frac{K_1+K_2}{M}>\omega_i^2, i\in\{1,2$$

Now, let us consider the third condition:

$$\underbrace{(\mathrm{K}_1+\mathrm{K}_2-\omega_2^2\mathrm{M})}_{\mathrm{C2}}\left[\underbrace{(\mathrm{K}_1+\mathrm{K}_2-\omega_1^2\mathrm{M})^2}_{\mathrm{C1}^2}+\underbrace{\omega_1^2\mathrm{B}^2}_{\mathrm{D1}}\right]$$
$$>\underbrace{(\mathrm{K}_1+\mathrm{K}_2-\omega_1^2\mathrm{M})}_{\mathrm{C1}}\left[\underbrace{(\mathrm{K}_1+\mathrm{K}_2-\omega_2^2\mathrm{M})^2}_{\mathrm{C2}^2}+\underbrace{\omega_2^2\mathrm{B}^2}_{\mathrm{D2}}\right]$$

And after short algebraic work out, we receive:

$$\mathrm{C1}+\frac{\mathrm{D1}}{\mathrm{C1}}>C2+\frac{\mathrm{D2}}{\mathrm{C2}}\to(\mathrm{K}_1+\mathrm{K}_2-\omega_1^2\mathrm{M})+\frac{\omega_1^2\mathrm{B}^2}{(\mathrm{K}_1+\mathrm{K}_2-\omega_1^2\mathrm{M})}$$
$$>(\mathrm{K}_1+\mathrm{K}_2-\omega_2^2\mathrm{M})+\frac{\omega_2^2\mathrm{B}^2}{(\mathrm{K}_1+\mathrm{K}_2-\omega_2^2\mathrm{M})}$$

$$(\omega_2^2-\omega_1^2)\left[\mathrm{M}+\frac{\mathrm{K}_1+\mathrm{K}_2}{(\mathrm{K}_1+\mathrm{K}_2-\omega_1^2\mathrm{M})(\mathrm{K}_1+\mathrm{K}_2-\omega_2^2\mathrm{M})}\right]>0$$

Since we determined $(\mathrm{K}_1+\mathrm{K}_2-\omega_\mathrm{i}^2\mathrm{M})>0$ as a preliminary condition, the above inequality is valid for $\omega_2^2>\omega_1^2$. Without loss of generality, we will assume $\omega_\mathrm{i}>0\ \forall i$, so that the following condition is equivalent to $\omega_2>\omega_1$. Therefore, summarizing the conditions:

$$\text{(7.3) } \omega_2>\omega_1 \text{ and } \frac{\mathrm{K}_1+\mathrm{K}_2}{\mathrm{M}}>\omega_\mathrm{i}^2, \mathrm{i}\in\{1,2\}$$

Another option to fulfill the inequality is:

$(\mathrm{K}_1+\mathrm{K}_2-\omega_2^2\mathrm{M})\alpha_1<0$ and

$(\mathrm{K}_1+\mathrm{K}_2-\omega_1^2\mathrm{M})\beta<0$ and

$(\mathrm{K}_1+\mathrm{K}_2-\omega_2^2\mathrm{M})\alpha_1<(\mathrm{K}_1+\mathrm{K}_2-\omega_1^2\mathrm{M})\beta$

This will lead to:

$$(\omega_2^2 - \omega_1^2)\left[M + \frac{K_1 + K_2}{(K_1 + K_2 - \omega_1^2 M)(K_1 + K_2 - \omega_2^2 M)}\right] < 0$$

Hence, the conditions in this case would be:

$$(7.4)\ \omega_1 > \omega_2 \text{ and } \frac{K_1 + K_2}{M} < \omega_i^2$$

Thus, enhanced sensitivity cannot be achieved unless one of the two following conditions is fulfilled:

- The frequency of the testing sine power source is higher than the frequency of the examined sine, and both frequencies are below the system's resonant frequency.

OR

- The frequency of the testing sine power source is lower than the frequency of the examined sine, and both frequencies are above the system's resonant frequency.

When the expression $\sum_{i=2}^{N} A_i \sin(\omega_i t)$ contains $\omega_i = \omega_1$

$$X_3 = \frac{1}{\sqrt{2}}\sqrt{\frac{\delta}{\alpha_1} + \sum_{i=3}^{N}\frac{A_i^2}{\alpha_i}}$$

The expression is very similar to the one we have seen in Section 5.2:

$$X_4 = \frac{1}{\sqrt{2}}\sqrt{\frac{A_1^2}{\alpha_1} + \sum_{i=2}^{N}\frac{A_i^2}{\alpha_i}}$$

Hence, the conditions could be derived easily from our calculations in the previous section (sensitivity to damping constant):

$$\sum_{i=3}^{N}\frac{A_i^2}{\alpha_i}\left(\frac{(K_1 + K_2 - \omega_i^2 M)\alpha_1}{(K_1 + K_2 - \omega_1^2 M)\alpha_i} - 1\right) > 0$$

Notice that the index begins with i=3 (instead of i=2, as in the previous section).

Sensitivity to damper constant

We will begin as before with the expressions:

$$X_1 = \frac{1}{\sqrt{2}} \frac{A}{\sqrt{(K - \omega_1^2 M)^2 + \omega_1^2 (B_1 + B_2)^2}}$$

And calculate the sensitivity:

$$S_{1\_B_1} = \left|\frac{\partial X_1}{\partial B_1} \cdot \frac{B_1}{X_1}\right| = \left|\frac{2A\omega_1^2 (B_1 + B_1)}{2\alpha^{\frac{3}{2}}} \cdot \frac{B_1 \alpha^{\frac{1}{2}}}{A}\right|$$

$$(7.5)\ S_{1\_B_1} = \frac{\omega_1^2 (B_1 + B_1) B_1}{\alpha}$$

Now, for the addition of a weighted sum of single sine power source ($\omega_i \neq \omega_1\ \forall\ i > 1$):

$$X_4 = \frac{1}{\sqrt{2}} \sqrt{\frac{A_1^2}{\alpha_1} + \sum_{i=2}^{N} \frac{A_i^2}{\alpha_i}}$$

$$S_7 = \left|\frac{\partial X_4}{\partial B_1} \cdot \frac{B_1}{X_4}\right| = \left|\frac{\frac{2A_1^2 \omega_1^2 (B_1 + B_2)}{\alpha_1^2} + \sum_{i=2}^{N} \frac{2A_i^2 \omega_i^2 (B_1 + B_2)}{\alpha_i^2}}{2\sqrt{\frac{A_1^2}{\alpha_1} + \sum_{i=2}^{N} \frac{A_i^2}{\alpha_i}}} \cdot \frac{B_1}{\sqrt{\frac{A_1^2}{\alpha_1} + \sum_{i=2}^{N} \frac{A_i^2}{\alpha_i}}}\right| =$$

$$= S_{1\_B_1} \frac{1 + \sum_{i=2}^{N} \frac{A_i^2 \omega_i^2 {\alpha_1}^2}{A_1^2 \omega_1^2 \alpha_i^2}}{1 + \sum_{i=2}^{N} \frac{\alpha_1 A_i^2}{\alpha_i A_1^2}}$$

The conditions for an improvement in the sensitivity are:

$$\sum_{i=2}^{N} \frac{A_i^2 \omega_i^2 {\alpha_1}^2}{A_1^2 \omega_1^2 \alpha_i^2} > \sum_{i=2}^{N} \frac{\alpha_1 A_i^2}{\alpha_i A_1^2}$$

$$(7.6)\sum_{i=2}^{N}\frac{A_i^2}{\alpha_i}(\frac{\alpha_1\omega_i^2}{\alpha_i\omega_1^2}-1)>0$$

And for the redundant version of this case:

$$\frac{\alpha_1\omega_2^2}{\beta\omega_1^2}>1$$

$$(7.7)\ \alpha_1\omega_2^2>\beta\omega_1^2$$

After some minor algebraic manipulations, we derive:

$$\omega_1{}^2\omega_2{}^2M^2>k^2\&\omega_1{}^2>\omega_2{}^2$$

or

$$\omega_1{}^2\omega_2{}^2M^2<k^2\&\omega_2{}^2>\omega_1{}^2$$

Finally, when the expression $\sum_{i=2}^{N}A_i\sin(\omega_i t)$ contains $\omega_i=\omega_1$, the expression we are analyzing is very similar to the one we have seen for $\omega_i\neq\omega_1$, and the condition for sensitivity improvement would be:

$$(7.8)\sum_{i=3}^{N}\frac{A_i^2}{\alpha_i}(\frac{\alpha_1\omega_i^2}{\alpha_i\omega_1^2}-1)>0$$

Sensitivity / Case	**Amp.**	**Freq.**	**Spring Const.**		**Damper Const.**
Probe -∿ **Test -∿**	X	X	$\omega_2{}^2 > \omega_1{}^2$ & $\frac{k_1+k_2}{M} < \omega_1{}^2$ Or $\frac{k_1+k_2}{M} > \omega_2{}^2$	$\omega_1{}^2 > \omega_2{}^2$ & $\frac{k_1+k_2}{M} < \omega_1{}^2$ & $\frac{k_1+k_2}{M} > \omega_2{}^2$	$\alpha\omega_2{}^2 > \beta\omega_1{}^2$ **Another option** $\omega_1{}^2\omega_2{}^2M^2 > k^2 \& \omega_1{}^2 > \omega_2{}^2$ Or $\omega_1{}^2\omega_2{}^2M^2 < k^2 \& \omega_2{}^2 > \omega_1{}^2$
Probe -∑ ∿ **Test -∿**	X	X	$\sum_{i=2}^{N}\frac{A_i^2}{\beta_i}\left\{\frac{[(k_1+k_2)-\omega_i{}^2M]}{\beta_i}-\frac{[(k_1+k_2)-\omega_1{}^2M]}{\alpha}\right\}>0$ (1)		$\sum_{i=2}^{N}\frac{A_i^2}{\beta_i}\left(\frac{\omega_i{}^2}{\beta_i}-\frac{\omega_1{}^2}{\alpha}\right) > 0$
Probe -∿ **Test -∿** (same ω)	X For all but $\mathrm{Cos}(\varphi) = -1$ And $A_1 = A_2$ For this condition: 0	0	0		0
Probe -∑ ∿ **Test –a∿** (same ω)	X	X	$\sum_{i=2}^{N}\frac{A_i^2}{\beta_i}\left\{\frac{[(k_1+k_2)-\omega_i{}^2M]}{\beta_i}-\frac{[(k_1+k_2)-\omega_1{}^2M]}{\alpha}\right\}>0$		$\sum_{i=2}^{N}\frac{A_i^2}{\beta_i}\left(\frac{\omega_i{}^2}{\beta_i}-\frac{\omega_1{}^2}{\alpha}\right) > 0$