

Minimum Acceleration Criterion with Constraints Implies Bang-Bang Control as an Underlying Principle for Optimal Trajectories of Arm Reaching Movements

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Rapid arm-reaching movements serve as an excellent test bed for any theory about trajectory formation. How are these movements planned? A minimum acceleration criterion has been examined in the past, and the solution obtained, based on the Euler-Poisson equation, failed to predict that the hand would begin and end the movement at rest (i.e., with zero acceleration). Therefore, this criterion was rejected in favor of the minimum jerk, which was proved to be successful in describing many features of human movements. This letter follows an alternative approach and solves the minimum acceleration problem with constraints using Pontryagin's minimum principle. We use the minimum principle to obtain minimum acceleration trajectories and use the jerk as a control signal. In order to find a solution that does not include nonphysiological impulse functions, constraints on the maximum and minimum jerk values are assumed. The analytical solution provides a three-phase piecewise constant jerk signal (bang-bang control) where the magnitude of the jerk and the two switching times depend on the magnitude of the maximum and minimum available jerk values. This result fits the observed trajectories of reaching movements and takes into account both the extrinsic coordinates and the muscle limitations in a single framework. The minimum acceleration with constraints principle is discussed as a unifying approach for many observations about the neural control of movements.

1 Introduction ---

The fast reaching movement is an elementary motor task mastered by humans and primates from an early age and is thought to be a primitive for more complex tasks (Morasso, 1981; Hogan, 1984; Gomi & Kawato, 1996; Karniel & Inbar, 1997; Smith, Brandt, & Shadmehr, 2000; Shadmehr & Wise,

2005). Rapid reaching movements are also called ballistic movements, since delays in the nervous system and in muscle activation prohibit the effective use of feedback in the first part of the movement, a phenomenon that was fruitfully employed to study the planning of movement and the capability of the brain to expect external perturbations (Shadmehr & Mussa-Ivaldi, 1994; Bhushan & Shadmehr, 1999; Karniel & Mussa-Ivaldi, 2003; Patton & Mussa-Ivaldi, 2004). Due to redundancy in the motor system, there are various ways to reach from one point in space to another utilizing different hand or joint trajectories and muscle activations (Bernstein, 1967). Experimental studies demonstrate that in typical conditions, point-to-point reaching movements are made using straight-line spatial trajectory and bell-shaped speed profiles (Morasso, 1981; Abend, Bizzi, & Morasso, 1982). How and why does the brain generate these well-observed invariant profiles?

The first question (the how) has been addressed by various computational models. Some of these models elaborate on the neural aspects of the system (Jordan, 1996; Barto, Fagg, Sitkoff, & Houk, 1999), while others focus on the muscle dynamics. Nonlinearities of the muscles and the spinal reflex loop may play a major role in generating the observed smooth bell-shaped speed profiles (Karniel & Inbar, 1997, 1999; Krylow & Rymer, 1997; Gribble, Ostry, Sanguineti, & Laboissiere, 1998). In order to answer the second question (the why), researchers usually assume that the biological system evolved to find optimal solutions, and under this approach, the question is, What is the cost function of the trajectory that is being minimized? A few approaches were proposed to determine this cost function. One approach is to minimize quantities that depend on the dynamics of the system. In this category, one can find minimum energy (Nelson, 1983), minimum torque change (Uno, Kawato, & Suzuki, 1989), and minimum isometric muscle torque change (Kashima & Isurugi, 1998). Another approach is to minimize quantities that depend on the kinematics of the system, that is, displacement and its derivatives in task-space coordinates (or angles and their derivatives in joints coordinates). In this category, one can find minimum acceleration, minimum jerk, minimum snap, and minimum crackle (Nelson, 1983; Flash & Hogan, 1985; Stein, Oguztoreli, & Capaday, 1986; Richardson & Flash, 2002; Dingwell, Mah, & Mussa-Ivaldi, 2004). A third approach suggests minimizing the errors caused by noise in the nervous system or environmental disturbances, which inserts noise to both the control signal and the feedback sensors (Harris & Wolpert, 1998).

The first approach considers physical quantities, such as energy and torque, directly. The third approach must refer to these physical quantities indirectly in order to compute the influence of the noise. Both approaches require complex calculations in order to find the optimal trajectory. With the second approach, where only the kinematics is considered, the optimal trajectory can be more easily derived analytically. Beyond this appeal to modelers, the second approach emphasizes the end point (the hand) as the

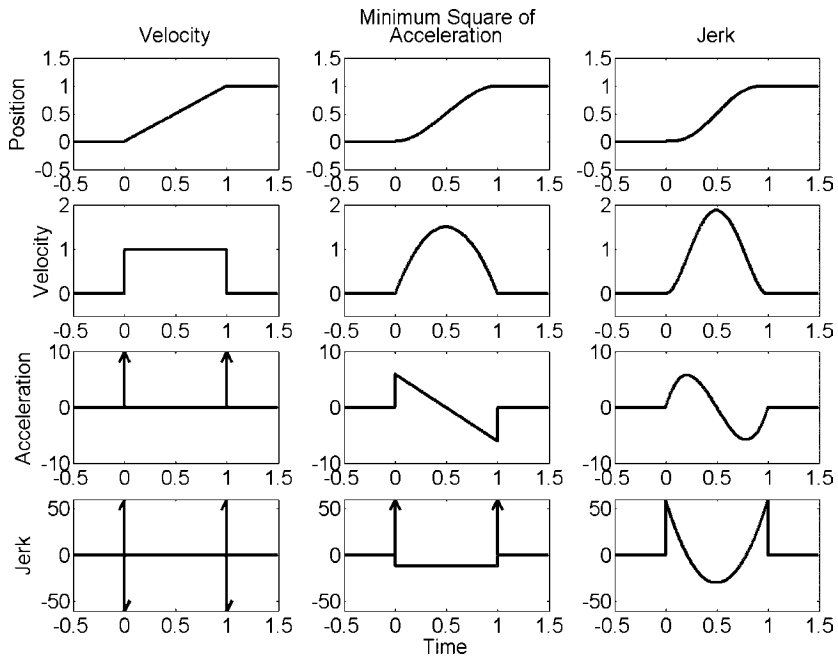


Figure 1: Position, velocity, acceleration, and jerk trajectories given by Euler-Poisson equation using different optimality criteria: minimum velocity (left column), minimum acceleration (middle column), and minimum jerk (right column). Units are normalized to the duration (T) and length (L) of the movement—speed in L/T , acceleration in L/T^2 , and jerk in L/T^3 .

relevant point for the optimization, in contrast to other approaches that consider joint space or muscle space.

Traditionally solutions to a minimum criterion that involve kinematics quantities were calculated analytically using the Euler-Poisson equation. Using an n th order derivative of the hands position as the optimum criterion, the Euler-Poisson equation, an ordinary differential equation, implies that the optimal trajectory is a $(2n - 1)$ th order polynomial function of time. The constants can be found by applying the boundary conditions. For example, minimum jerk with $n = 3$ has boundary conditions of zero velocity and zero acceleration at the start and the end of the movement (Richardson & Flash, 2002). Figure 1 depicts the speed profile predicted by this method. The analytical solution of the minimum acceleration criterion (MAC) shows no zero acceleration at the boundaries. This contradicts the observed hand rest in reaching movement before and after the movement; therefore, the MAC was rejected (Stein et al., 1986). The minimum jerk criterion (MJC) satisfies displacement, velocity, and acceleration boundary conditions, and

it became the most widespread criterion among the criteria of the second approach. A substantial body of research has demonstrated that this criterion can be used to explain a wide range of experimental data about human arm movements (Flash & Sejnowski, 2001; Sosnik, Hauptmann, Karni, & Flash, 2004).

In this study we propose a remedy to the MAC by adding acceleration boundary conditions and developing an analytical solution instead of changing the optimization criteria. We propose a new approach based on the Pontryagin minimum principle (Pontryagin, Boltyanskii, Gamkrelidze, & Mishchenko, 1962). In order to find a physiologically plausible solution, we also assumed constraints on the maximum and minimum jerk values, so we call our criterion a minimum acceleration criterion with constraints (MACC). The MACC predictions are compared with those of the MJC, and with experimental results. The derivation and proof of the optimal solution are difficult in the sense that they require modern analytical tools such as optimal control theory. However, the actual solution is quite simple—only two parameters should be determined in order to realize the optimal solution—and could be easily calculated by the nervous system in real time.

The rest of this letter is organized as follows. In the next section, we present the main theorem and analytically derive the trajectory predicted by the MACC. The solution is described rigorously and concisely, deferring the details to four appendixes. In section 3 we compare the predictions of the MACC to the predictions of the minimum jerk criterion. In section 4 we compare the predictions of the MACC to experimental results, and in section 5 we discuss the advantages and limitations of the MACC hypothesis.

2 Minimum Acceleration Criterion with Constraints

We strive to find the movement trajectory of the hand, which begins in a predefined start point and reaches a predefined end point at a predefined time. The beginning and end of the movement are defined by zero velocity and acceleration. Among all possible trajectories, we wish to find the trajectory that achieves a minimum mean-squared acceleration. In order to find a general solution, we employ tools from optimal control theory and formulate our task as a general optimal control problem. (For extended background on this theory, see one of the textbooks on optimal control theory: Kirk, 1970; Macki & Strauss, 1982; Lewis, 1992.)

In this formulation, a state vector, a control signal, and a cost function should be defined, and boundary conditions for the state vector, and an admissible control (i.e., limitations on the control signal) should be stated. The state vector is defined as the position of the hand and its first and second derivatives (velocity and acceleration, respectively). The control signal is defined as the end point jerk (third derivative). The cost function is described as the integral of the squared acceleration over the whole

movement. The limitations on the jerk are considered to be isomorphic (i.e., the maximum available jerk cannot exceed a certain amount, which is the same for each direction).

Hand-reaching movements are performed in the physical world in one, two, or three dimensions; however, the theorem that we present could also be applied to joint angles or other problems, and therefore we present the problem and the theorem in the general case. Then we show that the solution lies along a straight line (see lemma 1) and then present lemma 2, which is the one-dimension version of the theorem and could be used for most applications. Readers who are not interested in the general case and detailed proof may move to lemma 2, observe Figure 3, skip the proofs, and move to sections 3 and 4, which address only the particular solution presented in lemma 2 and Figure 3.

2.1 The Problem. In n -dimensional space, the state vector \underline{X} is defined as follows:

$$\underline{x}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}; \quad \underline{\dot{x}}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix}; \quad \underline{\ddot{x}}(t) = \begin{bmatrix} \ddot{x}_1(t) \\ \vdots \\ \ddot{x}_n(t) \end{bmatrix};$$

$$\underline{X} = \begin{bmatrix} \underline{x}(t) \\ \underline{\dot{x}}(t) \\ \underline{\ddot{x}}(t) \end{bmatrix}; \quad \underline{u}(t) = \begin{bmatrix} \ddot{x}_1(t) \\ \vdots \\ \ddot{x}_n(t) \end{bmatrix}$$

the system is described by

$$\underline{\dot{X}}(t) = \begin{bmatrix} \underline{\dot{x}}(t) \\ \underline{\ddot{x}}(t) \\ \underline{u}(t) \end{bmatrix} = \underline{f}(\underline{X}, \underline{u});$$

the boundary conditions are

$$\underline{x}(0) = \begin{bmatrix} x_{1_0} \\ \vdots \\ x_{n_0} \end{bmatrix}; \quad \underline{x}(T) = \begin{bmatrix} x_{1_f} \\ \vdots \\ x_{n_f} \end{bmatrix}$$

where $\sum_{i=1}^n (x_{i_f} - x_{i_0})^2 = L^2$;

$$\underline{\dot{x}}(0) = \underline{\ddot{x}}(0) = \underline{\dot{x}}(T) = \underline{\ddot{x}}(T) = \underline{0}$$

and the cost function is:

$$J(\underline{\mathbf{X}}, \underline{\mathbf{u}}) = \frac{1}{2} \int_0^T \sum_{i=1}^n \dot{x}_i^2(t) dt = \frac{1}{2} \int_0^T \mathbf{f}_0(\underline{\mathbf{X}}(t), \underline{\mathbf{u}}(t)) dt.$$

Here vectors are written in bold and underlined, T is the movement duration, and L is the movement length, $L \geq 0, T \geq 0$. The constraint on the jerk value is defined by $\sum_{i=1}^n u_i^2(t) \leq u_m$. Then, among all the trajectories that satisfy these conditions, we searched for the trajectory that minimizes the cost function and obtained the following solution.

Theorem 1. *The solution to the above problem is a straight line of the following form,*

$$\underline{\mathbf{x}}(t) = \underline{\mathbf{x}}(0) + (1/L)r(t) \cdot (\underline{\mathbf{x}}(T) - \underline{\mathbf{x}}(0)),$$

where $\underline{\mathbf{x}}(0)$ and $\underline{\mathbf{x}}(T)$ are the initial point and end point of the movement, respectively, and $r(t)$ is a time-dependent function consisting of three segments of third-order polynomials,

$$r(t) = \begin{cases} \frac{1}{6}u_m t^3 & 0 \leq t \leq t_1 \\ \frac{1}{6}c_0 t^3 - \frac{1}{2}c_1 t^2 + c_2 t + c_3 & t_1 \leq t \leq t_2, \\ \frac{1}{6}u_m t^3 - \frac{1}{2}u_m T \cdot t^2 + \frac{1}{2}u_m T^2 \cdot t - \frac{1}{6}u_m T^3 + L & t_2 \leq t \leq T \end{cases}$$

where

$$\begin{aligned} c_0 &= \frac{-24u_m \cdot L}{u_m \cdot T^3 - 24 \cdot L + \sqrt{u_m \cdot T^3(u_m \cdot T^3 - 24 \cdot L)}} \\ c_1 &= \frac{-12u_m \cdot L \cdot T}{u_m \cdot T^3 - 24 \cdot L + \sqrt{u_m \cdot T^3(u_m \cdot T^3 - 24 \cdot L)}} \\ c_2 &= \frac{(12 \cdot L - u_m \cdot T^3)\sqrt{u_m \cdot T} + u_m \cdot T^2 \sqrt{u_m \cdot T^3 - 24 \cdot L}}{4\sqrt{u_m \cdot T^3 - 24 \cdot L}} \\ c_3 &= \frac{(6 \cdot L - u_m \cdot T^3)\sqrt{u_m \cdot T^3 - 24 \cdot L} + (u_m \cdot T^3 - 18 \cdot L)\sqrt{u_m \cdot T^3}}{12\sqrt{u_m \cdot T^3 - 24 \cdot L}} \end{aligned}$$

and

$$t_1 = \frac{T}{2} \left(1 - \sqrt{\frac{u_m \cdot T^3 - 24 \cdot L}{u_m \cdot T^3}} \right)$$

$$t_2 = \frac{T}{2} \left(1 + \sqrt{\frac{u_m \cdot T^3 - 24 \cdot L}{u_m \cdot T^3}} \right).$$

Note that the position trajectory lies along the straight line that connects the initial and final hand positions.

Proof of Theorem 1. We first prove that the solution lies along the straight line (see lemma 1) and thus reduce the problem to a one-dimensional problem. Then we find the solution to this one-dimensional case (see lemma 2, which is proved by means of Pontryagin’s theorem and a few additional lemmas as described below).

Lemma 1. *The solution to the above n-dimensional problem is a straight line path that connects the initial point and end point of the hand movement.*

Proof of Lemma 1. Without loss of generality (by translating and rotating the axes), we can set the movement to begin at the origin and end at $\underline{x}(T) = [L \ 0 \ \dots \ 0]^T$. Let us assume any initial solution that is not a straight line. Then the acceleration vector $\ddot{\underline{x}}(t) = [\ddot{x}_1(t), \dots, \ddot{x}_n(t)]^T$ contains some nonnegative terms in the second, third, or n th dimensions at least part of the time. Now we build a new solution in which $\ddot{\underline{x}}(t) = [\ddot{x}_1(t), 0, \dots, 0]^T$. This solution consists of a straight line trajectory. It is clear that in this solution, the value of $J(\underline{\tilde{X}}, \underline{\tilde{u}})$ is not greater than the value in the initial (not a straight line) solution since

$$J(\underline{\tilde{X}}, \underline{\tilde{u}}) = \frac{1}{2} \int_0^T \sum_{i=1}^n \ddot{x}_i^2(t) dt$$

$$= \frac{1}{2} \int_0^T \ddot{x}_1^2(t) dt \leq \frac{1}{2} \int_0^T \sum_{i=1}^n \ddot{x}_i^2(t) dt = J(\underline{X}, \underline{u}).$$

In addition, the boundary conditions are still satisfied, because in the second, third, or n th dimensions,

$$\dot{\tilde{x}}_i(t) = \int_0^t \ddot{\tilde{x}}_i(\tau) d\tau = \int_0^t 0 d\tau = 0 \quad i = 2, 3, \dots, n$$

$$\tilde{x}_i(t) = \int_0^t \dot{\tilde{x}}_i(\tau) d\tau = \int_0^t 0 d\tau = 0 \quad i = 2, 3, \dots, n,$$

and in the first dimension, the straight line solution is the same as the curved trajectory solution:

$$\begin{aligned} \dot{x}_1(T) &= \int_0^T \ddot{x}_1(\tau) d\tau = \int_0^T \ddot{x}_1(\tau) d\tau = 0 \\ \bar{x}_1(T) &= \int_0^T \dot{x}_1(\tau) d\tau = \int_0^T \dot{x}_1(\tau) d\tau = L. \end{aligned}$$

The jerk maximum value constraint is also satisfied because

$$\begin{aligned} \tilde{\mathbf{u}}(t) &= \begin{bmatrix} \ddot{x}_1(t) \\ \vdots \\ \ddot{x}_n(t) \end{bmatrix} = \begin{bmatrix} u_1(t) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\ \sum_{i=1}^n \tilde{u}_i^2(t) &= u_1^2(t) \leq \sum_{i=1}^n u_i^2(t) \leq u_m. \end{aligned}$$

Therefore, for every curved trajectory, we can find a straight line trajectory that satisfies all the boundary conditions and the maximum jerk value constraint, with a value function that is not greater than that of the curved trajectory solution. This straight line goes along the path from initial point to end point of the movement. This is the end of the proof of lemma 1.

In lemma 1 we reduced the problem to a one-dimensional problem by showing that the trajectory lies on a straight line. Now we find the solution to the one-dimensional case, where the initial point of the movement is given on the origin.

Lemma 2. *From all the one-dimensional continuous and differentiable trajectories $x(t)$ with first and second continuous and differentiable derivatives, and bounded third derivative $-u_{m1} \leq \ddot{x}(t) \leq u_{m2}$, ($u_{m1}, u_{m2} > 0$) which satisfy the boundary conditions $x(0) = \dot{x}(0) = \ddot{x}(0) = \dot{x}(T) = \ddot{x}(T) = 0$, $x(T) = L$, the trajectory that minimizes the cost function $J = \frac{1}{2} \int_0^T \ddot{x}(t)^2 dt$ is a function consisting of three segments of third-order polynomials:*

$$x(t) = \begin{cases} \frac{1}{6}u_{m2}t^3 & 0 \leq t \leq t_1 \\ \frac{1}{6}c_0t^3 - \frac{1}{2}c_1t^2 + c_2t + c_3 & t_1 \leq t \leq t_2 \\ \frac{1}{6}u_{m2}t^3 - \frac{1}{2}u_{m2}T \cdot t^2 + \frac{1}{2}u_{m2}T^2 \cdot t + L - \frac{1}{6}u_{m2}T^3 & t_2 \leq t \leq T, \end{cases}$$

where the coefficients of the polynomial in the intermediate segment are

$$c_0 = \frac{-24 u_{m_2} \cdot L}{u_{m_2} \cdot T^3 - 24 \cdot L + \sqrt{u_{m_2} \cdot T^3 (u_{m_2} \cdot T^3 - 24 \cdot L)}}$$

$$c_1 = \frac{-12 u_{m_2} \cdot L \cdot T}{u_{m_2} \cdot T^3 - 24 \cdot L + \sqrt{u_{m_2} \cdot T^3 (u_{m_2} \cdot T^3 - 24 \cdot L)}}$$

$$c_2 = \frac{(12 \cdot L - u_{m_2} \cdot T^3) \sqrt{u_{m_2} \cdot T} + u_{m_2} \cdot T^2 \sqrt{u_{m_2} \cdot T^3 - 24 \cdot L}}{4 \sqrt{u_{m_2} \cdot T^3 - 24 \cdot L}}$$

$$c_3 = \frac{(6 \cdot L - u_{m_2} \cdot T^3) \sqrt{u_{m_2} \cdot T^3 - 24 \cdot L} + (u_{m_2} \cdot T^3 - 18 \cdot L) \sqrt{u_{m_2} \cdot T^3}}{12 \sqrt{u_{m_2} \cdot T^3 - 24 \cdot L}}$$

and the switching times are

$$t_1 = \frac{T}{2} \left(1 - \sqrt{\frac{u_{m_2} \cdot T^3 - 24 \cdot L}{u_{m_2} \cdot T^3}} \right)$$

$$t_2 = \frac{T}{2} \left(1 + \sqrt{\frac{u_{m_2} \cdot T^3 - 24 \cdot L}{u_{m_2} \cdot T^3}} \right).$$

Note that to obtain real numbers for the switching times, the maximum admissible jerk must satisfy $u_{m_2} \geq \frac{24 \cdot L}{T^3}$ and since c_0 is the jerk of the intermediate segment, the minimum admissible jerk must satisfy $-u_{m_1} \leq c_0$; otherwise, there is no solution to the control problem, that is, the control signal is not strong enough to bring the hand to the desired final state in the desired time.

Proof of Lemma 2. The following proof is the main thrust of this study, and it is based on Pontryagin's minimum principle, which specifies a condition that must be obeyed by the optimal solution. (Readers are referred to the optimal control literature for a more detailed description of this fundamental principle: Kirk, 1970; Macki & Strauss, 1982; Lewis, 1992.) Pontryagin's principle suggests that the optimal control is attained by finding the control signal, $u(t)$, that minimizes the function H , which is known in the literature as the Hamiltonian. We consider all the possible trajectories and find the optimal solution by means of contradiction (*reductio ad absurdum*) as the only solution that satisfies Pontryagin's minimum principle. We first define the state vector and control signal, cite Pontryagin's minimum principle, and formulate our problem in the notations of Pontryagin. Then we describe the nature of all possible solutions and narrow the space of possible solutions until we reach the only possible solution. In order to simplify the presentation of the proof, we divide this procedure into three lemmas and defer the details of their proof to the appendixes.

Let us define a state vector as the one-dimensional displacement, velocity, and acceleration of the trajectory, and the control signal as the jerk (i.e., third derivative of displacement), and obtain the system dynamics as follows:

$$\mathbf{X} = \begin{bmatrix} x(t) \\ \dot{x}(t) \\ \ddot{x}(t) \end{bmatrix}, \quad u(t) = \ddot{\ddot{x}}(t), \quad \dot{\mathbf{X}} = \begin{bmatrix} \dot{x}(t) \\ \ddot{x}(t) \\ u(t) \end{bmatrix} = f(\mathbf{X}, u).$$

*Pontryagin's minimum principle (Pontryagin et al., 1962).*¹ Consider the following system,

$$\dot{\mathbf{X}} = f(\mathbf{X}, u),$$

with an admissible control, $\mathbf{u}_{\min} \leq \mathbf{u} \leq \mathbf{u}_{\max}$, and the constraints $\mathbf{x}(0) = \mathbf{x}_0$; $\mathbf{x}(t_f) = \mathbf{x}_f$, and consider a cost function $J = \int_0^{t_f} f_0(\mathbf{x}(t), \mathbf{u}(t), t) dt$.

In addition let us define a function H of the variables $\mathbf{X}, u, \mathbf{P}, p_0$ (the Hamiltonian),

$$H(\mathbf{X}, u, \mathbf{P}, p_0) = p_0 f_0 + \mathbf{P}^T f(\mathbf{X}, u),$$

where the parameters vector \mathbf{P} satisfy:

$$\dot{\mathbf{P}}_i = -\frac{\partial H}{\partial \mathbf{X}_i}.$$

Now let \mathbf{u} be an admissible control, so that the solution $x_{[0,t_f]}$ provides the boundary conditions. In order that $\mathbf{u}(t)$ yield a solution of the given optimal problem with fixed time, it is necessary that there exists a nonzero continuous vector function $\mathbf{P}(t)$ corresponding to the function $\mathbf{u}(t)$ and $\mathbf{X}(t)$ such that:

1. For all $\{t : 0 < t < t_f\}$, the function H of the variable \mathbf{u} attains its minimum at the point $\mathbf{u} = \mathbf{u}(t)$.
2. The function $p_0(t)$ is nonnegative (and constant)

Proof. See Pontryagin et al. (1962, Chap. 2).

Following the notation of Pontryagin's minimum principle, since our cost function is $J = \int_0^{t_f} f_0(\mathbf{x}(t), \mathbf{u}(t), t) dt = \frac{1}{2} \int_0^1 \ddot{x}(t)^2 dt$, we can write the

¹In this textbook, the principle is described as a maximum principle rather than a minimum principle. The only difference in this case is that the function $p_0(t)$ is nonpositive, so the value function achieves its maximum.

Hamiltonian as follows:

$$H(u) = \frac{1}{2} p_0 \ddot{x}^2 + p_1 \dot{x} + p_2 \ddot{x} + p_3 u,$$

where $p_1(t)$, $p_2(t)$, $p_3(t)$ are continuous, differentiable, and bounded in $t \in [0, T]$, and comply with the time-differential equations:

$$\dot{p}_1(t) = 0$$

$$\dot{p}_2(t) = -p_1(t)$$

$$\dot{p}_3(t) = -p_0 \ddot{x} - p_2(t).$$

p_0 is a nonnegative constant and therefore can be considered as 0 or 1 (if $p_0 > 0$, then any value of p_0 only factorizes the function H by a constant, so without loss of generality, p_0 can be set to 1).

Pontryagin's minimum principle asserts that the optimal control signal (in our case, the jerk) must minimize the Hamiltonian. One should recall that according to our problem definition, the control signal is bounded $-u_{m1} \leq u(t) \leq u_{m2}$ and therefore our goal is to solve the following minimization problem:

$$\ddot{x}(t) = \underset{u \in [-u_{m1}, u_{m2}]}{\operatorname{arg\,min}} H(u).$$

To minimize the Hamiltonian ($H(u) \triangleq \frac{1}{2} p_0 \ddot{x}^2 + p_1 \dot{x} + p_2 \ddot{x} + p_3 u$), the control signal, $u(t)$, must be maximal and with sign opposite to $p_3(t)$. As a result, the control signal depends on the sign of $p_3(t)$, where $p_3(t)$ might change its sign over the time interval. In addition, there might be time intervals in which $p_3(t) = 0$ along the whole interval. In those intervals, the control signal does not affect the Hamiltonian directly, and other considerations should be taken in order to find the control signal. Therefore, we can divide the time interval into segments, where in each segment, $p_3(t)$ is positive, negative, or zero. If $p_3(t) \neq 0$ along the segment, the trajectory is called a nonsingular trajectory, and the control signal is maximal and with sign opposite to $p_3(t)$. If $p_3(t) = 0$, the trajectory along the segment is called a singular trajectory, and the control signal should be found indirectly. Each time point in which $p_3(t)$ changes its sign or changes from nonzero to zero or vice versa is called a switch. A general solution can include just a singular trajectory (i.e., $p_3(t) = 0$ all over the interval), just nonsingular trajectories (i.e., $p_3(t) \neq 0$ all over the interval but can change its sign), or a combination of singular and nonsingular trajectories (see Figure 2 for a demonstration of a combined solution).

To find the optimal solution we show by contradiction that no solution exists in the case of $p_0 = 0$ and that a singular trajectory must exist.

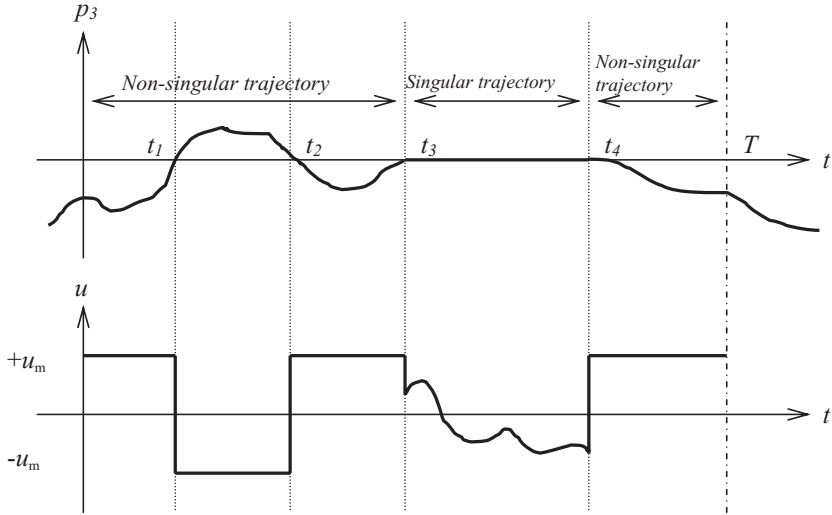


Figure 2: Combined solution that contains singular and nonsingular trajectories. If p_3 is nonzero, the control signal exceeds its maximum or minimum. If p_3 is zero, other considerations should be taken in order to determine the control signal.

Lemma 2.1. *There is no possible solution to the problem if $p_0 = 0$ (and therefore $p_0 > 0$).*

Proof. See appendix A.

Lemma 2.2. *The optimal trajectory must contain an intermediate singular trajectory.*

Proof. See appendix B.

Lemma 2.3. *The optimal trajectory must contain only two switches.*

Proof. See appendix C.

From lemma 2.3 we know that there are only three segments. From lemma 2.2, the intermediate segment must be singular trajectory; therefore, the solution consists of one singular trajectory that lies between two nonsingular trajectories. The three trajectories must form an overall continuous trajectory, with continuous first and second derivatives. In addition, the overall trajectory must satisfy the boundary conditions. Putting all these constraints together forms a system of 12 equations with 12 unknowns (see appendix D for details). In addition, the jerk in the first nonsingular trajectory and in the last nonsingular trajectory can be positive or negative. Each one of the four possibilities (positive or negative in the first and last nonsingular trajectories) gives a different solution, but not all of the solutions are physically acceptable. Solving the equations system four times with different choices of the jerk sign leads to the conclusion that only in the case of positive jerk in both the first and the last nonsingular trajectories

is a solution that satisfies $0 < t_1 < t_2 < T$. All other choices give switching times that are out of the movement time.

Solving the 12 equations system with the right choice of the sign of the jerk provides the switching times and the control signal in the singular trajectory (see appendix D for details):

$$\begin{aligned}
 t_1 &= \frac{T}{2} \left(1 - \sqrt{\frac{u_{m_2} \cdot T^3 - 24 \cdot L}{u_{m_2} \cdot T^3}} \right) \\
 t_2 &= \frac{T}{2} \left(1 + \sqrt{\frac{u_{m_2} \cdot T^3 - 24 \cdot L}{u_{m_2} \cdot T^3}} \right) \\
 c_0 &= \frac{-24 u_{m_2} \cdot L}{u_{m_2} \cdot T^3 - 24 \cdot L + \sqrt{u_{m_2} \cdot T^3 (u_{m_2} \cdot T^3 - 24 \cdot L)}}.
 \end{aligned}$$

There are two limitations that should be considered in examining this result. As we mentioned before, the switching times should be real numbers, which leads to the requirement that $u_{m_2} \geq \frac{24 \cdot L}{T^3}$; if the control signal is not strong enough, the trajectory cannot satisfy the boundary conditions. A second limitation is that the control signal in the singular trajectory should not exceed the boundary $c_0 \geq -u_{m_1}$. This limitation means that if the magnitude of the control signal in the nonsingular trajectories is too low, greater jerk must be used in the singular trajectory in order to compensate for the “slowness” of the nonsingular trajectories; otherwise the boundary conditions cannot be met.

The explicit solution of the optimal trajectory $x(t)$ is thus (see appendix D for details):

$$x(t) = \begin{cases} \frac{1}{6} u_{m_2} t^3 & 0 \leq t \leq t_1 \\ \frac{1}{6} c_0 t^3 - \frac{1}{2} c_1 t^2 + c_2 t + c_3 & t_1 \leq t \leq t_2 \\ \frac{1}{6} u_{m_2} t^3 - \frac{1}{2} u_{m_2} T \cdot t^2 + \frac{1}{2} u_{m_2} T^2 \cdot t + L - \frac{1}{6} u_{m_2} T^3 & t_2 \leq t \leq T \end{cases}$$

where

$$\begin{aligned}
 c_0 &= \frac{-24 u_{m_2} \cdot L}{u_{m_2} \cdot T^3 - 24 \cdot L + \sqrt{u_{m_2} \cdot T^3 (u_{m_2} \cdot T^3 - 24 \cdot L)}} \\
 c_1 &= \frac{-12 u_{m_2} \cdot L \cdot T}{u_{m_2} \cdot T^3 - 24 \cdot L + \sqrt{u_{m_2} \cdot T^3 (u_{m_2} \cdot T^3 - 24 \cdot L)}}
 \end{aligned}$$

$$c_2 = \frac{(12 \cdot L - u_{m_2} \cdot T^3) \sqrt{u_{m_2} \cdot T} + u_{m_2} \cdot T^2 \sqrt{u_{m_2} \cdot T^3 - 24 \cdot L}}{4 \sqrt{u_{m_2} \cdot T^3 - 24 \cdot L}}$$

$$c_3 = \frac{(6 \cdot L - u_{m_2} \cdot T^3) \sqrt{u_{m_2} \cdot T^3 - 24 \cdot L} + (u_{m_2} \cdot T^3 - 18 \cdot L) \sqrt{u_{m_2} \cdot T^3}}{12 \sqrt{u_{m_2} \cdot T^3 - 24 \cdot L}}$$

This is the end of the proof of theorem 1.

Note that the solution depends on only the maximum admissible jerk. The minimum limit of the jerk influences only the limit of c_0 and the capability (or incapability) of completing the movement. It is also worth noting that the switching times are symmetrically arranged around the middle of the time interval, so learning the solution requires learning only the difference of switching times from the middle of the time interval.

Another observation is that in the limiting case of $u_{m_2} \rightarrow \infty$, the switching times approach the initial and final times ($t_1 \rightarrow 0$, $t_2 \rightarrow T$), and the result reduces to $x(t) = (-2\tau^3 + 3\tau^2) \cdot L$, $\tau = \frac{t}{T}$, which is identical to the solution of the minimum acceleration criterion that was previously obtained employing the Euler-Poisson equation with no constraints on the control signal (see Figure 1).

3 Minimum Jerk Versus Minimum Acceleration with Constraints

The minimum jerk criterion (MJC) successfully predicts many experimental studies (Flash & Hogan, 1985; Sosnik et al., 2004). Therefore, any proposed criteria should be able to generate similar predictions under the tested conditions. According to MJC, knowing the boundary conditions of displacement, velocity, and acceleration is sufficient for finding the trajectory that satisfies the criterion. The MACC, on the other hand, also needs to know the maximum and minimum values of the admissible control signal. These values are important since they affect the shape of the velocity profile and hence affect the agreement of the theoretical solution with experimental results. A low value of maximum admissible control causes the switching times to move toward the middle of the time interval, whereas a high value pushes the switching times away from the middle toward the boundaries of the time interval.

Figure 3 depicts some theoretical trajectories for different values of jerk limitations. As one can see, low values create a bell-shaped velocity profile, and high values make more of an arched shape. As this value approaches infinity, the solution converges to the classical minimum acceleration solution obtained from the Euler-Poisson equation.

A salient difference between the two models is the shape of the jerk profile. While the MJC attains a second-order polynomial control signal, the MACC obtains a piecewise constant control signal. The maximum value of the jerk according to the MJC is $60L/T^3$ (Flash & Hogan, 1985; Sosnik

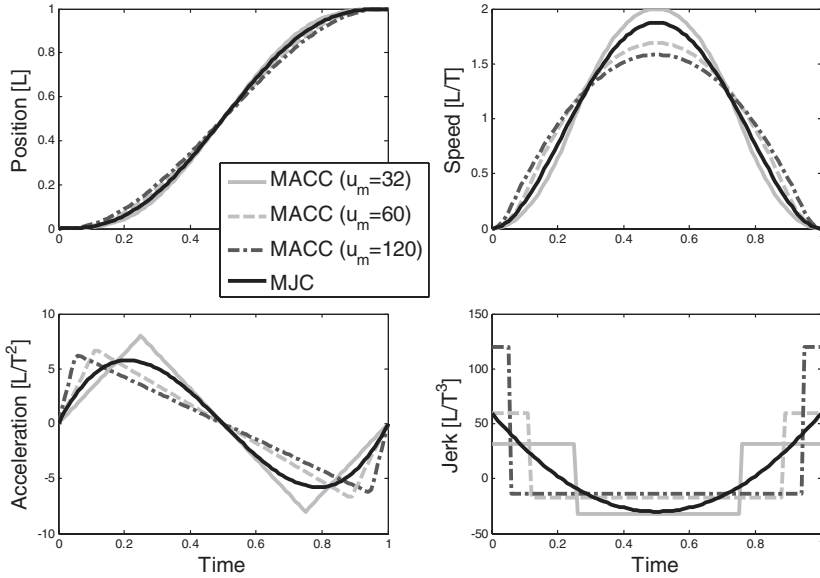


Figure 3: MACC versus MJC profiles. Displacement, speed, acceleration, and jerk trajectories of MACC solution with various jerk limitations. u_m is the maximum admissible jerk. Low admissible jerk (solid gray line) produces a bell-shaped speed profile. High admissible jerk (dash-dotted line) produces an arched-shaped speed profile. Medium admissible jerk (dashed line) produces an intermediate shape. Low admissible jerk speed profile is very close to the profile produced by MJC. Units are normalized to the duration (T) and length (L) of the movement: speed in L/T, acceleration in L/T², and jerk in L/T³.

et al., 2004) whereas by using MACC, the boundary conditions of displacement, velocity, and acceleration can be realized with almost half this value ($32L/T^3$).

It is also worthwhile to note that the maximum value of acceleration in the solution of MJC is lower than the maximum values obtained by the MACC. However, the overall mean squared acceleration is lower in the MACC than in the MJC when the maximum jerk is similar.

4 Fitting to Experimental Results

In this section we compare the theoretical prediction of the MACC hypothesis to human reaching movements obtained in previous studies (Karniel & Mussa-Ivaldi, 2002, 2003).

4.1 The Experiment. We present a brief description of the experiment and refer readers to Karniel and Mussa-Ivaldi (2002) for further details.

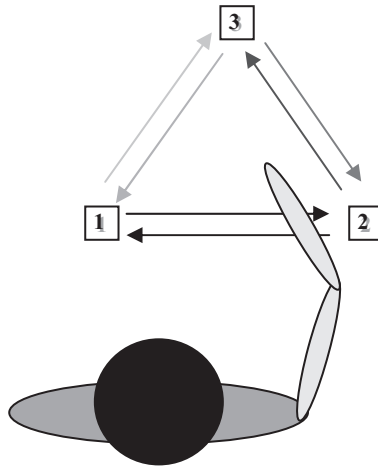


Figure 4: Experimental setup. One of three targets appeared on a screen at each moment triggering one of six possible movements of 10 cm.

Five subjects participated in the experiment. Seated subjects held the handle of a two degrees of freedom robotic manipulandum and looked at a screen that displayed the location of the hand and the location of the target. The movements were performed in the horizontal plane.

Subjects were asked to execute fast reaching movements to a target displayed on the screen. A small, round cursor represented the position of the hand, and a rectangular one represented the target. As soon as the cursor reached the target, the target either exploded (i.e., become gradually bigger over a period of 200 milliseconds) or changed color, instructing the subject to move faster or slower in order to achieve movement duration of approximately one-third of a second (± 50 milliseconds). The experiments had three possible targets, which allowed six possible movements of 10 cm (see Figure 4). Although the original experiments involved perturbing forces applied to the subjects' movements, here we analyze trials only from blocks in which no forces were applied. (For more details, see Karniel & Mussa-Ivaldi, 2002.)

4.2 Data Analysis. To analyze the raw data, the starting point and the stopping point should be found for each movement. This was done by examining the curvature of the position trajectory and the acceleration zero crossing. Before the initiation of the movement, the acceleration may be positive or negative, and the curvature could assume large values; however, once the movement starts, the acceleration is positive, and the movement becomes roughly straight. Therefore, we marked the starting at the first time where both conditions were met and hold until maximum velocity is

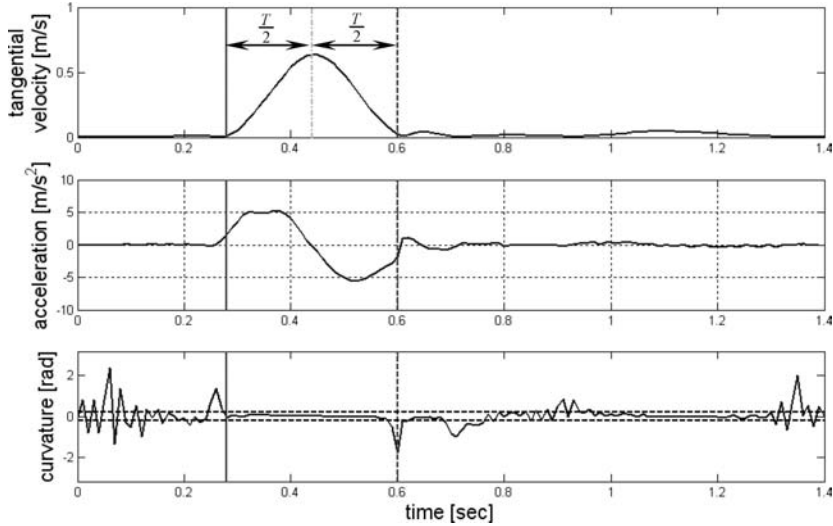


Figure 5: Tangential velocity profile, acceleration profile, and curvature of typical trajectory. Movement starts where acceleration is positive and curvature is greater than 0.2 [rad] (~ 11.5 [deg]) (left solid bar). Half time is set where the velocity is maximal (dash-dot bar). Trial duration is set to be twice the time to reach maximum velocity (right dashed bar).

obtained, specifically (1) the acceleration was no longer negative and (2) the curvature was less than 0.2 radians. The second condition was chosen by trial and error and sought to ensure a straight line with a reasonable tolerance. In order to avoid feedback effects in the last part of the movement, only the first half of the movement is analyzed. Since the duration of the movement depends on corrective movements that are not part of our model, we considered only the data at the beginning of the movement up to the point of maximum velocity. The time where the maximum velocity has been reached is considered the middle of the movement duration; thus, the overall duration of the movement is considered as twice the time to reach the maximum velocity. Figure 5 shows typical movements' velocity, acceleration, and curvature profiles and the starting and stopping points.

The MACC assumes maximum and minimum jerk value, but the existence of such constraints and their values is not known. In order to test the MACC, we fit the value of the maximum available jerk (u_{m_2}) to each trial by minimizing the mean squared error (MSE) of the tangential velocity profile. The fitting was performed only to the first half of the movement in order to avoid effects of feedback. Figure 6 shows the fitting to one typical trial. As one can see, the position and velocity profiles of MJC as well as MACC qualitatively match the experimental data. Outliers were considered as trials where the calculated u_{m_2} was less than $27 \frac{L}{T^3}$ (in this case $u_{m_1} > 2u_{m_2}$) or

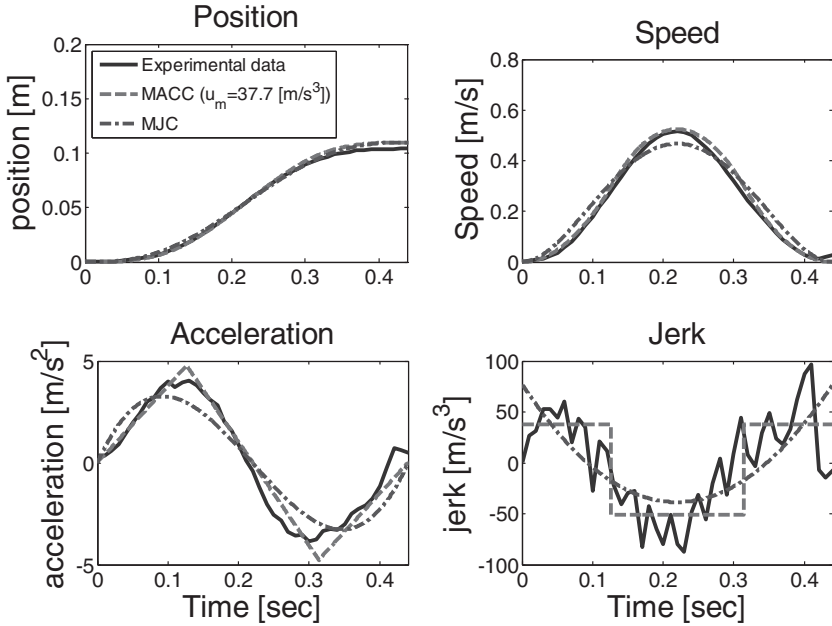


Figure 6: Trajectories of position, velocity, acceleration and jerk. Experimental result (solid), MACC (dashed), and MJC (dash-dotted).

greater than $65 \frac{L}{T^3}$ (twice the median) and were removed from the data set used in further analysis. (Since the duration and length of movements vary between directions and subjects, the value of u_{m_2} was normalized by length and time and was expressed by means of $\frac{L}{T^3}$ units.)

After outline removal, we were left with 80, 85, 79, 71, and 77 trials for subjects 1, 2, 3, 4, and 5, respectively. These trials were used for the data analysis presented in the next section.

4.3 Comparing the Fitting of MJC and MACC to the Data. We used two measures to compare the predictions of the MJC and those of the MACC to the data; the expected maximum velocity and the mean error over the trajectory profile (see Figure 7).

From the analytical results of MACC (see lemma 2), one can derive the maximum velocity as a function of u_{m_2} :

$$V_{\max} = \frac{1}{8} \left(T^2 u_{m_2} - \sqrt{T u_{m_2} (T^3 u_{m_2} - 24L)} \right).$$

The maximum velocity estimated by MJC is $1.875 \frac{L}{T}$ (Richardson & Flash, 2002).

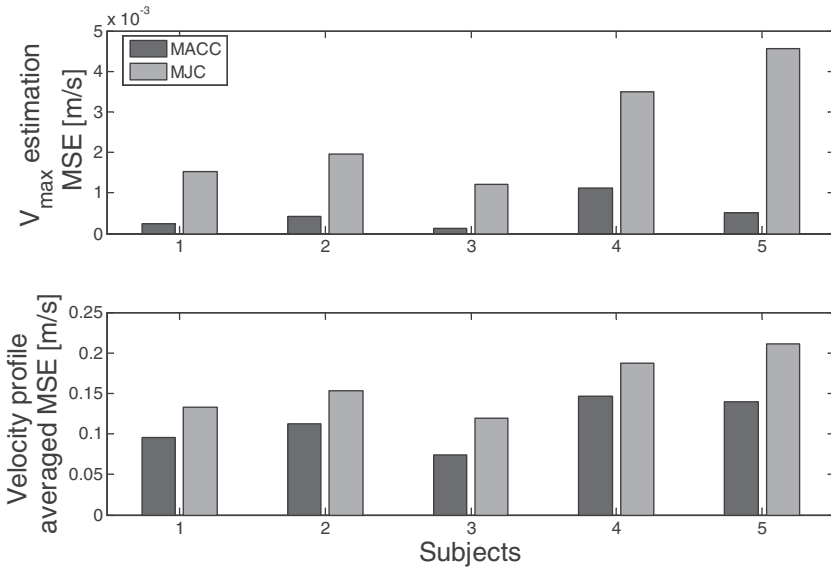


Figure 7: Average of MSE between experimental results and analytic results of MACC and MJC. MSE of V_{\max} (top). MSE of velocity profile (Bottom). One can see that the MACC can fit the experimental data better than the MJC can. The differences between the errors in both measures for each subject were statistically significant ($p < 0.01$).

The difference between MACC estimation and the data was calculated and compared with the difference between MJC estimation and the data. In addition, the sum of squared error in the velocity trajectory between analytic results and experimental data was calculated in every trial (from the initiation of movement up to the point of maximum velocity) for both MJC and MACC. This analysis shows that the MACC can fit the experimental data better than the MJC (see Figure 7). The differences between the errors in both measures for each subject were statistically significant ($p < 0.01$ using Wilcoxon rank sum test).

4.4 The Maximum Jerk at Different Movement Directions. The maximum jerk, u_{m_2} , was fitted to each movement. Although subjects received feedback encouraging them to perform movements of the same duration, the time duration of each trial varied by 12% to 18% of median value. The changes in time duration play an important role in determining the control signal, since the jerk depends on time duration by power of three. Therefore, it is difficult to analyze the properties of u_{m_2} , which also changes from trial to trial. Nevertheless, one can still observe differences in the values of the maximum jerk that depend on the direction of the movements. We

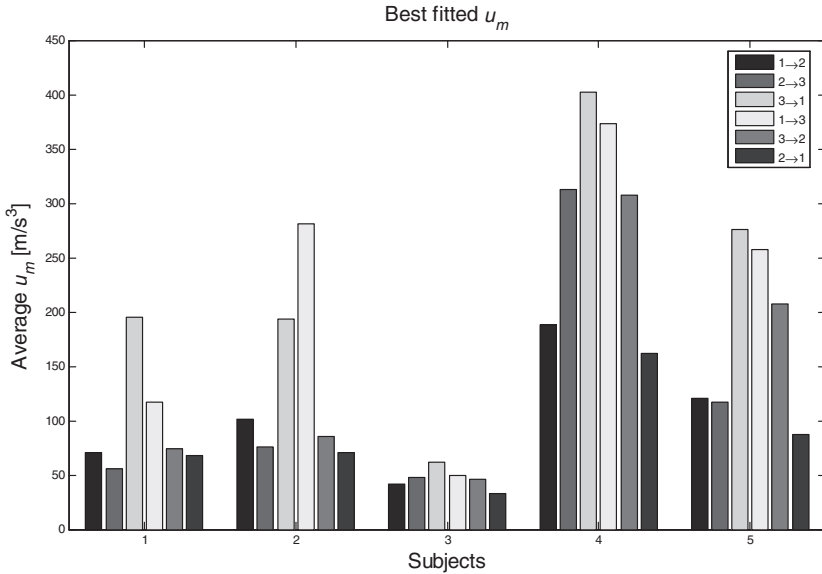


Figure 8: Average of u_m for six directions in five subjects. Movements from target 1 to target 3 and vice versa have the greatest u_m . It is interesting to note that movements between targets 1 and 3 require much less shoulder movement.

have calculated the mean of the maximum jerk for the six directions (see Figure 4) for each one of the five subjects (see Figure 8). One can see that in four of five subjects, the maximum jerk value was clearly greater in movements between targets 1 and 3 and targets 3 and 1, movements that do not include the shoulder; relative to the other movements that involve shoulder movement. These differences are statistically significant ($p < 0.05$ using the Wilcoxon rank sum test) only in subjects 1 and 2. This result indicates that the maximum available jerk depends on the muscle and joint properties, and therefore the MACC is not a pure extrinsic criterion.

5 Discussion

We present a new criterion for arm trajectory formation, the minimum acceleration with constraints criterion (MACC); analytically derive the expected hand trajectories; and compare them to the minimum jerk criterion as well as to measured hand movements of five subjects.

The solution of the MACC dictates a simple three-phase control signal where the controller should provide only two parameters: the switching times (due to symmetry, one parameter is sufficient) and the value of the jerk of the singular trajectory. This control could be easily learned by trial and error.

The analytical solution assumes an existence of a maximum available jerk value. From our experimental results, it is clear that there is no such unique value, and this value changes from one movement to another. Our physiological interpretation of the bound on the jerk could be related to the neural signal integrated before the force is produced and therefore could be similar to jerk; another interpretation is the desire to minimize wear and tear that might increase significantly above a certain value of jerk. We already noticed that the maximum value changes with movement type consistently, which indicates that the true maximum jerk could be calculated in the joint coordinates. It is not clear that the maximum available jerk value stays constant over one movement, and if this is the case, the solution presented here should be considered as an approximation of the underlying mechanism, which should be further studied. If the admissible jerk control does not stay constant over the whole movement, the analytical solution becomes more complicated, but the policy of finding the optimal control according to the sign of p_3 remains unchanged in the sense that this work could be extended to consider various policies to determine the maximum available jerk at each movement.

We have clearly found that the MACC better fits the experimental data than the MJC for all subjects (see Figure 7); nevertheless, one should remember that the MACC includes an additional free parameter: the maximum available jerk. As modelers look for simplicity, this could be considered a disadvantage, but considering the physiological interpretation, this additional flexibility can indirectly introduce the constraints imposed by the muscles and joints, and in that sense, the MACC can enjoy the advantages of the two schools of criteria that concentrate on either the joints or the end point (see Figure 8). In order to find out whether MACC or MJC is more compliant with experimental data, more accurate measures of acceleration and jerk are required. One of the predictions that could be used to distinguish between the two criteria is the arch-shaped velocity profile for large, admissible jerk values. If we assume that the maximum admissible jerk is constant (does not depend on the length and duration of the movement), then differences in length of movement or in time duration are expected to change the normalized maximum admissible jerk and, consequently, the velocity profile. In our experimental data, all the movements were approximately equal in length and time duration. Additional experiments should be done with changes in length and duration in order to check whether there is a change in velocity profile. Another prediction involves the maximum jerk value, which could be rather low in the MACC and in the extreme case almost half of the maximum jerk expected from the MJC; accurate measurements of the maximum jerk under various movement directions and muscle conditions could reveal the potential flexibility of the maximum jerk parameters facilitated by the MACC.

Further study is required to thoroughly compare the MACC predictions to the many other alternative criteria partly presented in section 1. In this

letter, we concentrated on comparison to the minimum jerk criterion due to its simple analytical solution; the other methods require assumptions as to the limb dynamics, muscle properties, or noise distributions, and therefore a simple, objective comparison is quite difficult. Nevertheless, it is worth noting that the free parameter inherent in the MACC (the maximum jerk) provides a bridge to dynamic related criteria since it accounts for the different profiles observed in different arm configurations and movement directions (see Figure 8) as predicted by muscle- and joint-related criteria. One prominent prediction of many of the other criteria is a gently curved profile rather than a straight line, which is predicted by both MJC and the MACC (see, e.g., Harris & Wolpert, 1998; Uno et al., 1989). In this respect, it is interesting to note our simplifying assumptions about the value of the maximum jerk being similar in all direction of movements. An interesting direction for future study is the generalization of the model to the possibility of different values of the maximum jerk in different movement directions, which may imply a nonstraight path as observed in some cases, in particular in 3D and at the edge of the work space.

Dingwell et al. (2004) and recently Svinin, Goncharenko, Zhi-Wei, and Hosoe (2006) tried to predict the trajectory of a hand holding a spring attached to a mass. Following the tradition of using the Euler-Poisson equation, which was used to dismiss the minimum acceleration criterion in favor of the minimum jerk criterion, Dingwell et al. claim that since the solution of MJC on the mass gives a nonzero acceleration of the hand, a higher derivative of the mass displacement should be taken as the minimum criterion, and a minimum crackle criterion was suggested. According to our approach, using the minimum principle can solve the problem of hand attached to a mass with a spring using MJC or even using the minimum acceleration, but constraints should be added to the admissible control. Furthermore, if a jerk of the hand is considered as the control signal, it is not difficult to show that the control signal is piecewise constant, that is, a bang-bang control. (However, further study is required in order to find the number of switches and magnitude of jerk in each segment.)

The bang-bang control method was proposed in the context of minimum time, which predicts maximum velocity and nonsmooth trajectories, predictions that were rightfully criticized as not physiological. In the original minimum acceleration criterion (see Figure 1), the acceleration changes abruptly, and therefore the predicted trajectories significantly differ from the observed hand movements. It is important to note that here, the bang-bang control is in the jerk signal, and under the proposed MACC model, both position and velocity are smooth, and even the acceleration is continuous. The predicted acceleration profile is not smooth, however; abrupt changes in the jerk are also predicted by the MJC and should not be considered not physiological as they represent neural control signal, which is being integrated by the musculoskeletal system to generate smooth movement. The neural control signal, measured as a firing rate, does not have

to be smooth, and burst activities are frequently observed in the nervous system; see, for example, the bistability recently observed in the cerebellum *in vivo* (Loewenstein et al., 2005).

The MACC predicts bang-bang control and suggests that the brain may control only the transitions rather than sending a continuous command. This approach is consistent with the notion of hierarchy, in which the brain sends simple commands that are further translated to movement by the spinal cord muscles and arm dynamics. This can be traced back to William Harvey,² who used the analogy of an army to describe the control of movements. More recent studies use the notion of primitives to describe this notion (Mussa-Ivaldi & Bizzi, 2000), in one case even suggesting that the brain minimizes the transitions in its command in order to simplify its efforts for frequently used movements (Karniel, Mussa-Ivaldi, d'Avella, & Bizzi, 2002). This hypothesis, named the minimum transition hypothesis, is based on the notion of intermittence control implied by the MACC presented in this letter.

Woodworth (1899) discussed the distinction between initial adjustment and current control and inspired many modern studies about intermittence control and the notion of generating submovements as a result of feedback about error during the movement (Hanneton, Berthoz, Droulez, & Slotine, 1997; Doeringer & Hogan, 1998; Novak, Miller, & Houk, 2002; Fishbach, Roy, Bastianen, Miller, & Houk, 2005). Although the MACC do not address the feedback at all, the control strategy predicted by the MACC, of using pulses and steps, perfectly fits with this notion of intermittence control and could serve as the basis for a more general view of the motor control system. Such simple motor commands of pulses and steps were used in a few models that describe neural control of movement (Karniel & Inbar, 1997; Barto et al., 1999). It is also interesting to note that recent measurements in the cerebellum found clear evidence for an intermittence control strategy (Loewenstein et al., 2005). Whether the brain employs continuous or discrete control strategy remains a fascinating open question. In this study, we show that a discrete control strategy could be consistent with the observed smooth movements that were previously explained by means of continuous control strategies.

An analytical solution for the optimal hand trajectory under MACC was derived. This new criterion predicts a simple trajectory of the three-phase bang-bang control strategy. The bang-bang control strategy is the optimal result of many control problems, as described in any textbook on optimal control (Kirk, 1970; Macki & Strauss, 1982; Lewis, 1992), and it is fruitfully used in many systems, from missile control to domestic thermostat. Further

²Harvey (1578–1657) wrote, “Nature sets in motion by signs and watchwords, which are made with little momentum. . . . Just as in the army the soldiers are set in motion by one word as if by a given signal and continue to move until they receive another signal to stop, so the muscles move in order and harmony from established custom” (Harvey, W. 1959).

studies are required to derive predictions for other types of movements, such as reaching through via points, and to test the predictions of this model by accurately measuring jerk during reaching movements.

All in all, it appears that the rumors about the death of the minimum acceleration criterion were premature.

Appendix A: Proof of Lemma 2

In this appendix, we prove lemma 2.1 by contradiction, that is, we show that there is no possible solution to the problem if $p_0 = 0$.

Assume $p_0 = 0$. Then the Hamiltonian is

$$H = p_1\dot{x} + p_2\ddot{x} + p_3u,$$

and

$$\dot{p}_1(t) = 0 \quad \Rightarrow \quad p_1(t) = c_0$$

$$\dot{p}_2(t) = -p_1(t) \Rightarrow p_2(t) = -c_0t + c_1$$

$$\dot{p}_3(t) = -p_2(t) \Rightarrow p_3(t) = \frac{1}{2}c_0t^2 - c_1t + c_2.$$

First, let us consider the case where $p_3 \equiv 0$ in a segment (subinterval) $[t_1, t_2] \in [0, T]$. (If $p_3 = 0$ only in a finite number of time points (singular time points), it does not affect the system dynamics.) Since $p_1(t), p_2(t), p_3(t)$ do not depend on $x(t)$ or its derivatives, c_0, c_1, c_2 stay constant over the whole time interval $t \in [0, T]$. Because $p_3 \equiv 0$ over a segment $[t_1, t_2] \in [0, T]$, all its derivatives in the segment are also equal to zero:

$$p_3 = \frac{1}{2}c_0t^2 - c_1t + c_2 = 0 \Rightarrow c_2 = 0$$

$$\dot{p}_3 = -p_2 = c_0t - c_1 = 0 \Rightarrow c_1 = 0$$

$$\ddot{p}_3 = -\dot{p}_2 = p_1 = c_0 = 0 \Rightarrow c_0 = 0.$$

But this leads to the fact that the vector $\mathbf{P} \triangleq [p_0, p_1(t), p_2(t), p_3(t)]^T$ is identically zero, which contradicts the requirement in theorem 1 that \mathbf{P} is a nonzero vector. Therefore, there is no segment $[t_1, t_2] \in [0, T]$, in which $p_3 \equiv 0$.

Now, let us consider the case in which $p_3(t) \neq 0$ on any segment in $t \in [0, T]$. The function H is linear by u , so its minimum is attained at the boundaries of $u(t)$, that is,

$$u(t) = \arg \min_{-u_{m_1} \leq u \leq u_{m_2}} H(u) = -\text{sgn}(p_3(t)).$$

u_{\max} , where $u_{\max} = \begin{cases} u_{m_2} & \text{sgn}(p_3(t)) < 0 \\ u_{m_1} & \text{sgn}(p_3(t)) > 0 \end{cases}$, and u_{m_1}, u_{m_2} limit the admissible control (jerk of $x(t)$).

Since the control signal depends on $p_3(t)$ and $p_3(t)$ might change its sign over the time interval, one can divide the time interval into segments, where in each segment, $p_3(t)$ is positive or negative along the whole segment (see Figure 2).

In each segment, the solution is of the form

$$\begin{aligned} u_i &= -\operatorname{sgn}(p_3(t)) u_{\max} \\ \ddot{x} &= -\operatorname{sgn}(p_3(t)) u_{\max} t + {}^i d_0 \\ \dot{x} &= -\frac{1}{2} \operatorname{sgn}(p_3(t)) u_{\max} t^2 + {}^i d_0 t + {}^i d_1 \\ x &= -\frac{1}{6} \operatorname{sgn}(p_3(t)) u_{\max} t^3 + \frac{1}{2} {}^i d_0 t^2 + {}^i d_1 t + {}^i d_2, \end{aligned}$$

where i is the index of the segments and is defined as the number of switches in the control signal before that segment (i.e., i is zero in the first segment). The continuity of $x(t)$ and its first and second derivatives dictates the number of the segments. The boundary conditions, as well as the continuity of $x(t)$ and its first and second derivatives, give the following constraints:

$$\begin{aligned} x(0) &= 0 & x(T) &= L & x(t_i^-) &= x(t_i^+) \\ \dot{x}(0) &= 0 & \dot{x}(T) &= 0 & \dot{x}(t_i^-) &= \dot{x}(t_i^+) \\ \ddot{x}(0) &= 0 & \ddot{x}(T) &= 0 & \ddot{x}(t_i^-) &= \ddot{x}(t_i^+), \end{aligned}$$

where t_i , $i = 1..k$ are the switching times and k is the number of switches. The total number of equations is $6 + 3k$. The variables to be found are:

$$\begin{array}{ll} {}^i d_0, {}^i d_1, {}^i d_2 & i = 0..k \\ t_i & i = 1..k. \end{array}$$

The total number of variables to be found is $3(k + 1) + k$. A general solution might exist only if the total number of variables to be found is equal to or greater than the total number of equations. This gives us a lower boundary to the number of switches:

$$\begin{aligned} 6 + 3k &\leq 3(k + 1) + k \\ 6 + 3k &\leq 4k + 3 \\ k &\geq 3. \end{aligned}$$

One can see that the number of switches cannot be less than three.

In addition, other constraints should be considered. The Lagrangian multiplier $p_3(t)$ must be zero in all switching times:

$$p_3(t_i) = \frac{1}{2}c_0t_i^2 - c_1t_i + c_2 = 0 \quad i = 1..k$$

But $p_3(t)$ is a second-order polynomial, so it can equal zero at most twice; therefore, it is not possible to have three (or more) switching times. We conclude that no possible solution is applicable when $p_0 = 0$.

Appendix B: Proof of Lemma 2.2

In this appendix we prove lemma 2.2; we continue based on appendix A and show that the optimal trajectory must contain an intermediate singular trajectory.

The conclusion of lemma 2.1 is that $p_0 = 1$. The problem is now the same as defined in appendix A, except that the Hamiltonian is defined as

$$H(u) \triangleq \frac{1}{2}\dot{x}^2 + p_1\dot{x} + p_2\ddot{x} + p_3u,$$

and the third element of the vector P is defined by

$$\dot{p}_3(t) = -\ddot{x} - p_2(t).$$

By integration, the Lagrangian multipliers are

$$p_1(t) = c_0$$

$$p_2(t) = -c_0t + c_1$$

$$p_3(t) = -\dot{x} + \frac{1}{2}c_0t^2 - c_1t + c_2.$$

Notice that $p_1(t)$, $p_2(t)$ are independent in $x(t)$, so c_0 , c_1 are constants over the whole interval $t \in [0, T]$. In addition, the first derivative of $x(t)$ is continuous, and $p_3(t)$ is continuous too, which requires that c_2 is also constant over the whole interval $t \in [0, T]$.

As in appendix A, one can see that since the control signal depends on the sign of $p_3(t)$ and $p_3(t)$ might change its sign over the time interval, the time interval can be divided into segments, where in each segment $p_3(t)$ is positive, negative, or zero along the whole segment (see Figure 2).

Assume (by contradiction) that there are no singular trajectories along the whole interval $t \in [0, T]$. Then the whole interval contains only nonsingular trajectories, and $p_3(t) \neq 0$, which yields $\ddot{x}(t) = -\text{sgn} p_3(t) \cdot u_{\max}$, where

$u_{\max} = \begin{cases} u_{m2} \operatorname{sgn}(p_3(t)) < 0 \\ u_{m1} \operatorname{sgn}(p_3(t)) > 0 \end{cases}$ and u_{m1}, u_{m2} limit the admissible control (jerk of $x(t)$).

The form of $x(t)$ and its first and second derivatives in a single segment is as follows:

$$\begin{aligned} \ddot{x}(t) &= -\operatorname{sgn}(p_3(t)) u_{\max} \\ \dot{x}(t) &= -\operatorname{sgn}(p_3(t)) u_{\max} t + {}^i d_0 \\ x(t) &= -\frac{1}{2} \operatorname{sgn}(p_3(t)) u_{\max} t^2 + {}^i d_0 t + {}^i d_1 \\ x(t) &= -\frac{1}{6} \operatorname{sgn}(p_3(t)) u_{\max} t^3 + \frac{1}{2} {}^i d_0 t^2 + {}^i d_1 t + {}^i d_2, \end{aligned}$$

where i indicates the index of the segment (starting from 0), $\{{}^i d_j\}_{j=0}^2$ are constants, and u_{\max} as above. The continuity constraints on $x(t)$ and its first and second derivatives dictate the number of segments. Adding the boundary conditions, gives the following constraints:

$$\begin{aligned} x(0) = 0 \quad x(T) = L \quad x(t_i^-) = x(t_i^+) \\ \dot{x}(0) = 0 \quad \dot{x}(T) = 0 \quad \dot{x}(t_i^-) = \dot{x}(t_i^+) \\ \ddot{x}(0) = 0 \quad \ddot{x}(T) = 0 \quad \ddot{x}(t_i^-) = \ddot{x}(t_i^+), \end{aligned}$$

where $t_i, i = 1..k$ are the switching times and k is the number of switches. The total number of equations equals $6 + 3k$.

The variables to be found are:

$$\begin{aligned} {}^i d_0, {}^i d_1, {}^i d_2 \quad i = 0..k \\ t_i \quad i = 1..k. \end{aligned}$$

The total number of variables to be found is $3(k + 1) + k$. A general solution might exist only if the total number of variables to be found is equal to or greater than the total number of equations. This gives us a lower boundary to the number of switches:

$$\begin{aligned} 6 + 3k &\leq 3(k + 1) + k \\ 6 + 3k &\leq 4k + 3 \\ k &\geq 3. \end{aligned}$$

One can see that the number of switches cannot be fewer than three, so the number of segments in the total interval cannot be fewer than four. In addition, other constraints should be considered. The Lagrangian multiplier $p_3(t)$ must be zero in all switching times:

$$\begin{aligned} p_3(t_i) = \frac{1}{2}(c_0 + \operatorname{sgn}(p_3(t_i^-)) \cdot u_{\max}) t_i^2 - (c_1 + {}^{i-1} d_0) t_i + (c_2 - {}^{i-1} d_1) = 0 \\ i = 1..k. \end{aligned}$$

Some of the d_j variables were found by the continuity and boundary constraints. The number of yet undetermined variables equals the difference between the number of variables ($4k + 3$) and the number of equations ($6 + 3k$), that is, $(k - 3)$.

Adding the three variables c_0, c_1, c_2 , brings the total number of yet undetermined variables to k . The number of equations derived from the constraints on $p_3(t)$ is also k . This concludes that if k is equal to or greater than three, the number of equations equals the number of variables, and thus finite number of solutions to the problem may exist.

Let us take a close look on one intermediate segment in $t \in [t_1, t_k]$ (i.e., any segment except for the first one or the last one). $p_3(t)$ is a second-order polynomial, with roots on t_i, t_{i+1} . If $p_3(t)$ is negative, its second derivative is positive, which leads to the following result:

$$\begin{aligned} p_3(t) &< 0 \\ \Downarrow \\ \frac{1}{2}(c_0 + \text{sgn}(p_3(t)) \cdot u_{m2}) &> 0 \\ c_0 - u_{m2} &> 0 \\ u_{m2} &< c_0. \end{aligned}$$

If $p_3(t)$ is negative, its second derivative is positive, and hence

$$\begin{aligned} p_3(t) &> 0 \\ \Downarrow \\ \frac{1}{2}(c_0 + \text{sgn}(p_3(t)) \cdot u_{m1}) &< 0 \\ c_0 + u_{m1} &< 0 \\ -u_{m1} &> c_0. \end{aligned}$$

Since we assume that none of the segments contains singular trajectories, any two adjacent internal segments must have an opposite sign of $p_3(t)$. This imposes that $u_{m2} < c_0$ and $u_{m1} < -c_0$, which means that either $u_{m1} < 0$ or $u_{m2} < 0$, but u_{m1}, u_{m2} are both positive by definition. The only way that avoids the contradiction is when all the internal nonsingular trajectories have the same sign, and singular trajectories separate them. This eliminates the possibility of a solution with no singular trajectory segment.

Appendix C: Proof of Lemma 2.3

In this appendix we prove lemma 2.3, that is, we show that the optimal trajectory contains only two switches.

In appendix B we saw that the optimal trajectory must contain at least one singular trajectory. In a singular trajectory, $p_3(t) = 0$ along a segment. This yields that all of its derivatives in the segment become zero, which results in

$$\begin{aligned}\ddot{p}_3(t) &= -\ddot{x} - \dot{p}_2(t) = -\ddot{x} + p_1(t) = -\ddot{x} + c_0 = 0 \\ \ddot{x}(t) &= c_0.\end{aligned}$$

Hence, the control signal, $u(t) = \ddot{x}(t)$, has constant values along segments in the time interval, and its form is of switching between these constant values. This implies also that $p_3(t)$ switches between second-order polynomials.

Integrating three times gives the form of $x(t)$ in a singular trajectory:

$$x(t) = \frac{1}{6}c_0t^3 - \frac{1}{2}c_1t^2 + c_2t + c_3.$$

In the presence of singular trajectories, the continuity constraints are as follows:

$$\begin{aligned}x(0) &= 0 & x(T) &= L & x(t_i^-) &= x(t_i^+) \\ \dot{x}(0) &= 0 & \dot{x}(T) &= 0 & \dot{x}(t_i^-) &= \dot{x}(t_i^+) \\ \ddot{x}(0) &= 0 & \ddot{x}(T) &= 0 & \ddot{x}(t_i^-) &= \ddot{x}(t_i^+).\end{aligned}$$

This yields $6 + 3k$ equations, where k is the number of switches, but there are fewer variables, since c_0, c_1, c_2 are the same in all the singular trajectories. The variables are:

$$\begin{aligned}i d_0, i d_1, i d_2 & \quad i = 0..k \text{ and segment } i \text{ is not a singular one} \\ c_0, c_1, c_2, i c_3 & \quad i = 0..k \text{ and segment } i \text{ is a singular one} \\ t_i & \quad i = 1..k\end{aligned}$$

The number of parameters to be determined is $4k - 2m + 6$, where m is the number of singular trajectories. Here, the lower boundary of number of switches is defined by

$$6 + 3k \leq 4k - 2m + 6$$

$$k \geq 2m.$$

From the above inequality, one can see that the number of switches has to be equal to or greater than two (since $m \geq 1$), and the number of segments must be equal to or greater than three.

Now assume (by contradiction) that there are more than three segments. If so, there exists at least one internal nonsingular trajectory. If $p_3(t)$ is

negative in that segment, then

$$\begin{aligned}
 p_3(t) &< 0 \\
 \Downarrow \\
 \frac{1}{2}(c_0 + \operatorname{sgn}(p_3(t)) \cdot u_{m2}) &> 0 \\
 c_0 - u_{m2} &> 0 \\
 u_{m2} &< c_0.
 \end{aligned}$$

Since c_0 is the third derivative of $x(t)$ in the singular trajectory, this contradicts the control problem definition, which bounds the jerk by u_{m2} .

If $p_3(t)$ is positive in that segment, then

$$\begin{aligned}
 p_3(t) &> 0 \\
 \Downarrow \\
 \frac{1}{2}(c_0 + \operatorname{sgn}(p_3(t)) \cdot u_{m1}) &< 0 \\
 c_0 + u_{m1} &< 0 \\
 -u_{m1} &> c_0,
 \end{aligned}$$

but this contradicts the lower bound of the jerk in the control problem definition. This implies that no nonsingular trajectories can be present in an internal segment, and as a consequence the number of switches is two.

Appendix D: System Equations and Boundary and Continuity Constraints Equations

In this appendix we provide the system equations and the boundary and continuity constraints equations in detail.

In appendix C we concluded that the trajectory consists of three segments. This yields two switch times in which continuity constraints on the displacement, velocity, and acceleration should be satisfied.

The control signal in the first and last segments can be u_{m2} or $-u_{m1}$, so there are four possibilities. In order to find the solution(s), we have to solve four equation systems. Only solutions with switching times in the interval $t \in [0, T]$ can be accepted. It is not difficult to find out that the only system that gives an acceptable solution is the one with positive control signal in the first and last segments.

The continuity constraints on displacement are

$$\begin{aligned}
 \frac{1}{6}u_{m2}t_1^3 + \frac{1}{2}{}^0d_0t_1^2 + {}^0d_1t_1 + {}^0d_2 &= \frac{1}{6}c_0t_1^3 + \frac{1}{2}c_1t_1^2 + c_2t_1 + c_3 \\
 \frac{1}{6}c_0t_2^3 + \frac{1}{2}c_1t_2^2 + c_2t_2 + c_3 &= \frac{1}{6}u_{m2}t_2^3 + \frac{1}{2}{}^2d_0t_2^2 + {}^2d_1t_2 + {}^2d_2,
 \end{aligned}$$

the continuity constraints on velocity are

$$\begin{aligned}\frac{1}{2}u_{m_2}t_1^2 + {}^0d_0t_1 + {}^0d_1 &= \frac{1}{2}c_0t_1^2 + c_1t_1 + c_2 \\ \frac{1}{2}c_0t_2^2 + c_1t_2 + c_2 &= \frac{1}{2}u_{m_2}t_2^2 + {}^2d_0t_2 + {}^2d_1,\end{aligned}$$

and the continuity constraints on acceleration are

$$\begin{aligned}u_{m_2}t_1 + {}^0d_0 &= c_0t_1^2 + c_1 \\ c_0t_2 + c_1 &= u_{m_2}t_2 + {}^2d_0.\end{aligned}$$

In addition, the boundary conditions should also be satisfied:

$$\begin{aligned}x(0) &= 0 & x(T) &= L \\ \dot{x}(0) &= 0 & \dot{x}(T) &= 0 \\ \ddot{x}(0) &= 0 & \ddot{x}(T) &= 0,\end{aligned}$$

where u_{m_1} , u_{m_2} , L , T are part of the problem definition, and 0d_0 , 0d_1 , 0d_2 , c_0 , c_1 , c_2 , c_3 , 2d_0 , 2d_1 , 2d_2 , t_1 , t_2 are the unknown parameters.

A system with 12 equations and 12 unknown parameters is obtained.

Solving this system yields the following results:

$$\begin{aligned}t_1 &= \frac{T}{2} \left(1 - \sqrt{\frac{u_{m_2} \cdot T^3 - 24 \cdot L}{u_{m_2} \cdot T^3}} \right); & t_2 &= \frac{T}{2} \left(1 + \sqrt{\frac{u_{m_2} \cdot T^3 - 24 \cdot L}{u_{m_2} \cdot T^3}} \right) \\ c_0 &= \frac{-24u_{m_2} \cdot L}{u_{m_2} \cdot T^3 - 24 \cdot L + \sqrt{u_{m_2} \cdot T^3}(u_{m_2} \cdot T^3 - 24 \cdot L)} \\ c_1 &= \frac{-12u_{m_2} \cdot L \cdot T}{u_{m_2} \cdot T^3 - 24 \cdot L + \sqrt{u_{m_2} \cdot T^3}(u_{m_2} \cdot T^3 - 24 \cdot L)} \\ c_2 &= \frac{(12 \cdot L - u_{m_2} \cdot T^3)\sqrt{u_{m_2} \cdot T} + u_{m_2} \cdot T^2\sqrt{u_{m_2} \cdot T^3 - 24 \cdot L}}{4\sqrt{u_{m_2} \cdot T^3 - 24 \cdot L}} \\ c_3 &= \frac{(6 \cdot L - u_{m_2} \cdot T^3)\sqrt{u_{m_2} \cdot T^3 - 24 \cdot L} + (u_{m_2} \cdot T^3 - 18 \cdot L)\sqrt{u_{m_2} \cdot T^3}}{12\sqrt{u_{m_2} \cdot T^3 - 24 \cdot L}} \\ {}^0d_0 &= {}^0d_1 = {}^0d_2 = 0 \\ {}^2d_0 &= -u_{m_2} \cdot T; & {}^2d_1 &= \frac{u_{m_2} \cdot T^2}{2}; & {}^2d_2 &= L - \frac{u_{m_2} \cdot T^3}{6}.\end{aligned}$$

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References

- Abend, W., Bizzi, E., & Morasso, P. (1982). Human arm trajectory formation. *Brain*, *105*, 331–348.
- Barto, A. G., Fagg, A. H., Sitkoff, N., & Houk, J. C. (1999). A cerebellar model of timing and prediction in the control of reaching. *Neural Computation*, *11*, 565–594.
- Bernstein, N. (1967). *The coordination and regulation of movements*. New York: Pergamon Press.
- Bhushan, N., & Shadmehr, R. (1999). Computational nature of human adaptive control during learning of reaching movements in force fields. *Biological Cybernetics*, *81*, 39–60.
- Dingwell, J. B., Mah, C. D., & Mussa-Ivaldi, F. A. (2004). Experimentally confirmed mathematical model for human control of a non-rigid object. *Journal of Neurophysiology*, *91*, 1158–1170.
- Doeringer, J. A., & Hogan, N. (1998). Intermittency in preplanned elbow movements persists in the absence of visual feedback. *Journal of Neurophysiology*, *80*, 1787–1799.
- Fishbach, A., Roy, S. A., Bastianen, C., Miller, L. E., & Houk, J. C. (2005). Kinematic properties of on-line error corrections in the monkey. *Experimental Brain Research*, *164*, 442–457.
- Flash, T., & Hogan, N. (1985). The coordination of arm movements: An experimentally confirmed mathematical model. *Journal of Neuroscience*, *5*, 1688–1703.
- Flash, T., & Sejnowski, T. J. (2001). Computational approaches to motor control. *Current Opinion in Neurobiology*, *11*, 655–662.
- Gomi, H., & Kawato, M. (1996). Equilibrium-point control hypothesis examined by measured arm stiffness during multijoint movement. *Science*, *272*, 117–120.
- Gribble, P. L., Ostry, D. J., Sanguineti, V., & Laboisiere, R. (1998). Are complex control signals required for human arm movement? *Journal of Neurophysiology*, *79*, 1409–1424.
- Hanneton, S., Berthoz, A., Droulez, J., & Slotine, J. J. E. (1997). Does the brain use sliding variables for the control of movements? *Biological Cybernetics*, *77*, 381–393.
- Harris, C. M., & Wolpert, D. M. (1998). Signal-dependent noise determines motor planning. *Nature*, *394*, 780–784.
- Harvey, W. (1959). *De moto locali animalium* [On animal movement]. (G. Whitteridge, Trans.). Cambridge: Cambridge University Press.
- Hogan, N. (1984). An organizing principle for a class of voluntary movements. *Journal of Neuroscience*, *4*, 2745–2754.
- Jordan, M. I. (1996). Computational aspects of motor control and motor learning. In H. Heuer, & S. Keele (Eds.), *Handbook of perception and action: Motor skills*. New York: Academic Press.
- Karniel, A., & Inbar, G. F. (1997). A model for learning human reaching movements. *Biological Cybernetics*, *77*, 173–183.

- Karniel, A., & Inbar, G. F. (1999). The use of a nonlinear muscle model in explaining the relationship between duration, amplitude, and peak velocity of human rapid movements. *Journal of Motor Behavior*, 31, 203–206.
- Karniel, A., & Mussa-Ivaldi, F. A. (2002). Does the motor control system use multiple models and context switching to cope with a variable environment? *Experimental Brain Research*, 143, 520–524.
- Karniel, A., & Mussa-Ivaldi, F. A. (2003). Sequence, time, or state representation: How does the motor control system adapt to variable environments? *Biological Cybernetics*, 89, 10–21.
- Karniel, A., Mussa-Ivaldi, F. A., d'Avella, A., & Bizzi, E. (2002). *A minimum transition hypothesis to account for motor synergies*. Abstract presented at the twelfth annual meeting of the neural control of movement meeting, Naples, FL.
- Kashima, T., & Isurugi, Y. (1998). Trajectory formation based on physiological characteristics of skeletal muscles. *Biological Cybernetics*, 78, 413–422.
- Kirk, D. E. (1970). *Optimal control theory: An introduction*. Upper Saddle River, NJ: Prentice Hall.
- Krylow, A. M., & Rymer, W. Z. (1997). Role of intrinsic muscle properties in producing smooth movements. *IEEE Transactions on Biomedical Engineering*, 44, 165–176.
- Lewis, F. L. (1992). *Applied optimal control and estimation*. Upper Saddle River, NJ: Prentice Hall.
- Loewenstein, Y., Mahon, S., Chadderton, P., Kitamura, K., Sompolinsky, H., Yarom, Y. et al. (2005). Bistability of cerebellar Purkinje cells modulated by sensory stimulation. *Nature Neuroscience*, 8, 202–211.
- Macki, J., & Strauss, A. (1982). *Introduction to optimal control theory*. New York: Springer-Verlag.
- Morasso, P. (1981). Spatial control of arm movements. *Experimental Brain Research*, 42, 223–227.
- Mussa-Ivaldi, F. A., & Bizzi, E. (2000). Motor learning through the combination of primitives. *Philosophical Transactions of the Royal Society of London Series B—Biological Sciences*, 355, 1755–1769.
- Nelson, W. L. (1983). Physical principles for economies of skilled movements. *Biological Cybernetics*, 46, 135–147.
- Novak, K. E., Miller, L. E., & Houk, J. C. (2002). The use of overlapping submovements in the control of rapid hand movements. *Experimental Brain Research*, 144, 351–364.
- Patton, J. L., & Mussa-Ivaldi, F. A. (2004). Robot-assisted adaptive training: Custom force fields for teaching movement patterns. *IEEE Transactions on Biomedical Engineering*, 51, 636–646.
- Pontryagin, L. S., Boltyanskii, V., Gamkrelidze, R., & Mishchenko, E. (1962). *The mathematical theory of optimal processes*. New York: Wiley.
- Richardson, M. J. E., & Flash, T. (2002). Comparing smooth arm movements with the two-thirds power law and the related segmented-control hypothesis. *Journal of Neuroscience*, 22, 8201–8211.
- Shadmehr, R., & Mussa-Ivaldi, F. A. (1994). Adaptive representation of dynamics during learning of a motor task. *Journal of Neuroscience*, 14, 3208–3224.
- Shadmehr, R., & Wise, S. P. (2005). *The computational neurobiology of reaching and pointing: A foundation for motor learning*. Cambridge, MA: MIT Press.

- Smith, M. A., Brandt, J., & Shadmehr, R. (2000). Motor disorder in Huntington's disease begins as a dysfunction in error feedback control. *Nature*, *403*, 544–549.
- Sosnik, R., Hauptmann, B., Karni, A., & Flash, T. (2004). When practice leads to co-articulation: The evolution of geometrically defined movement primitives. *Experimental Brain Research*, *156*, 422–438.
- Stein, R. B., Oguztoreli, M. N., & Capaday, C. (1986). What is optimized in muscular movements? In N. L. Jones, N. McCartney, & A. J. McComas (Eds.), *Human muscle power* (pp. 131–150). Champaign, IL: Human Kinetics.
- Svinin, M., Goncharenko, I., Zhi-Wei, L., & Hosoe, S. (2006). Reaching movements in dynamic environments: How do we move flexible objects? *IEEE Transactions on Robotics*, *22*, 724–739.
- Uno, Y., Kawato, M., & Suzuki, R. (1989). Formation and control of optimal trajectory in human multijoint arm movement: Minimum torque-change model. *Biological Cybernetics*, *61*, 89–101.
- Woodworth, R. S. (1899). The accuracy of voluntary movement. *Psychol. Rev.*, *3*, 1–114.

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