Nonlinear dynamics and pattern formation with applications to ecology

Ehud Meron
Ben Gurion University, Israel

Part III: Multimode localized structures

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Outline of part III

Background: Localized structures, multi-mode systems

Multi-mode localized structures – the general idea

Hopf-Turing systems:

1. The Hopf-Turing bifurcation in the FHN model
2. Amplitude equations for the Hopf-Turing bifurcation
3. Effects of spatial and temporal periodic forcing

Localized Turing structures hosting the Hopf mode:

1. Self-organized waveguides
2. Multi-state localized sturctures

Localized Hopf structures hosting the Turing mode:

Hosting phenomena in cusp-Hopf systems

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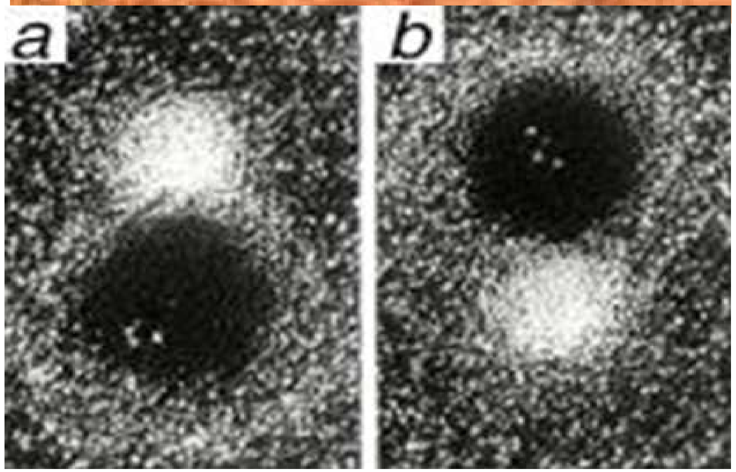
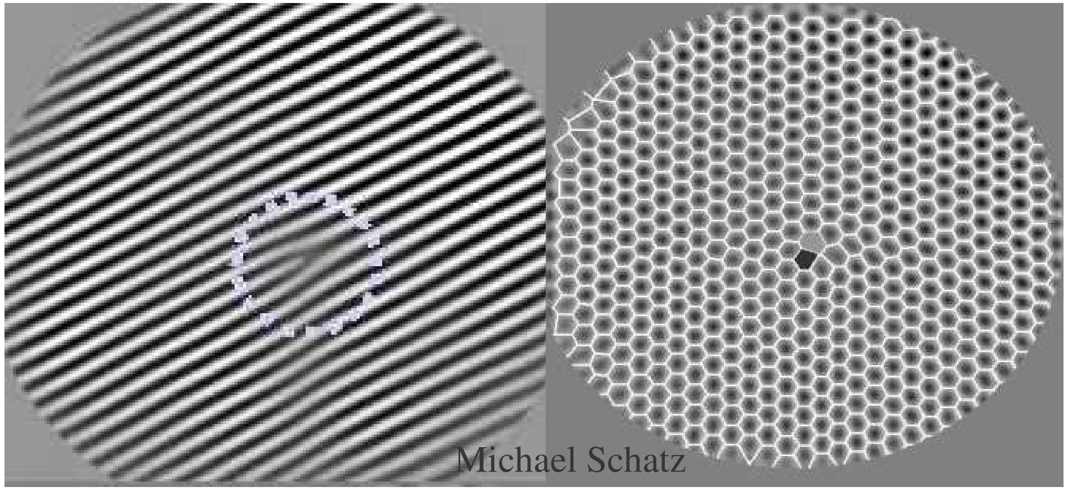
Localized sturctures

Oscillons in vibrated granular media

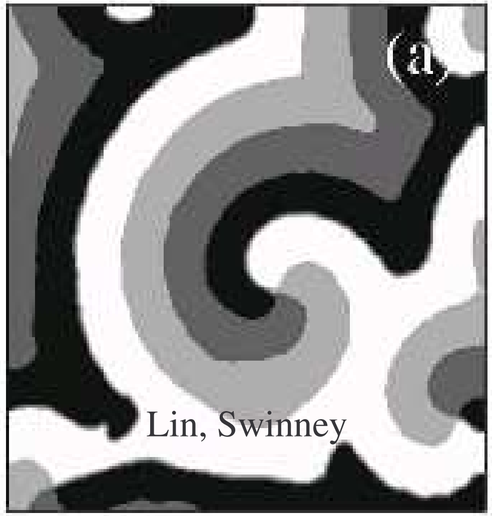


Dislocation and penta-hepta defects in periodic patterns



Oscillons in surface waves

Four-phase spirals in the BZ reaction



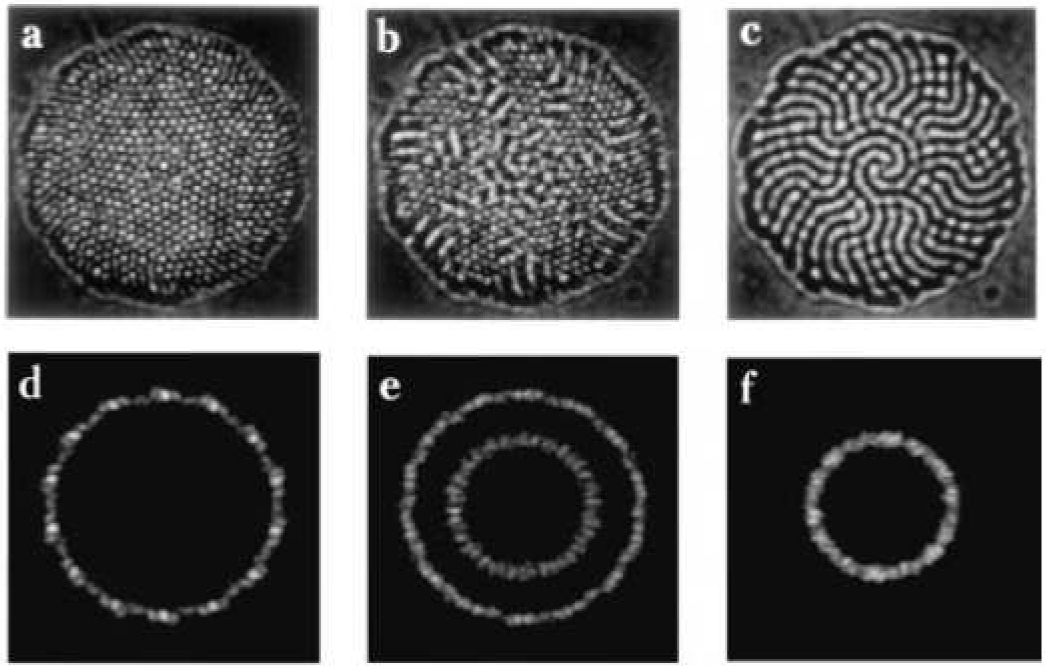
Multimode systems

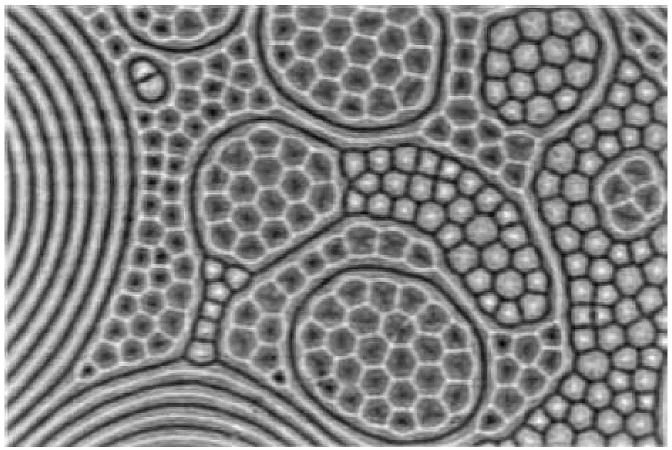
The localized structures shown above all involve a single mode, but spatially extended systems often give rise to multiplicity of modes.

The modes may appear simultaneously in multiple instabilities, or sequentially in a series of instabilities.

Different spatial modes in optical patterns

(Residori, Ramazza, Pampaloni, Boccaletti, Arecchi, PRL 1996).



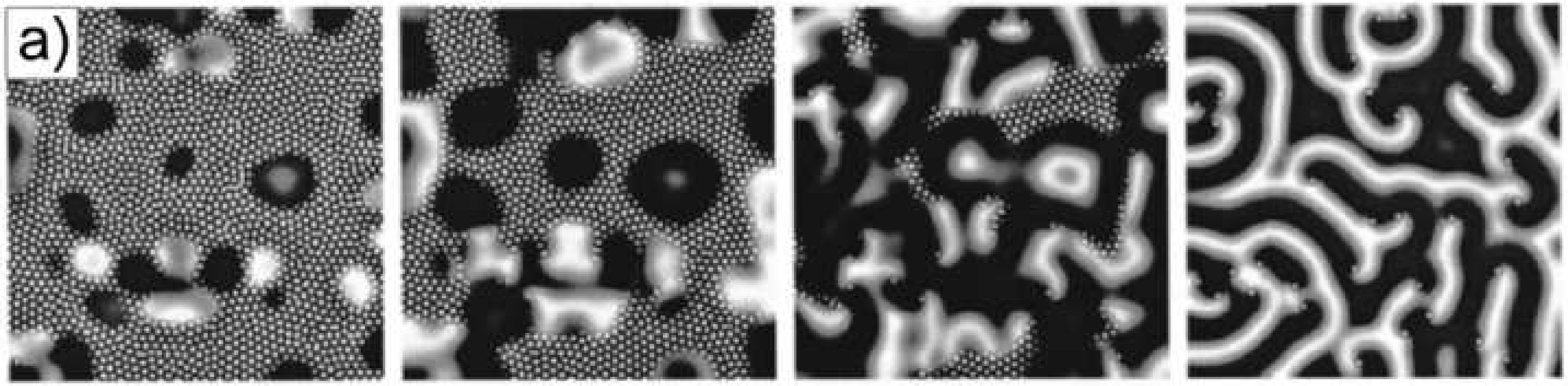


Roll and hexagon modes in thermal convection (Assenheimer and Steinberg, 1996).

Multimode localized structures

Quite often one mode non-linearly damps the others:

Hopf and Turing modes in an RD model (Yang, Dolnik, Zhabotinsky, Epstein, JCP 2002)

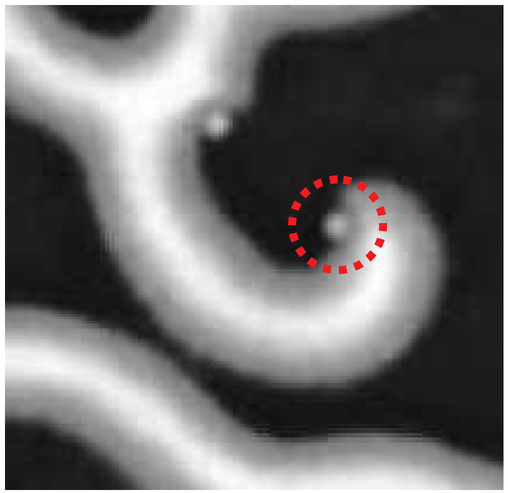


→ Time

The Hopf mode wins over the Turing mode as time proceeds,

BUT

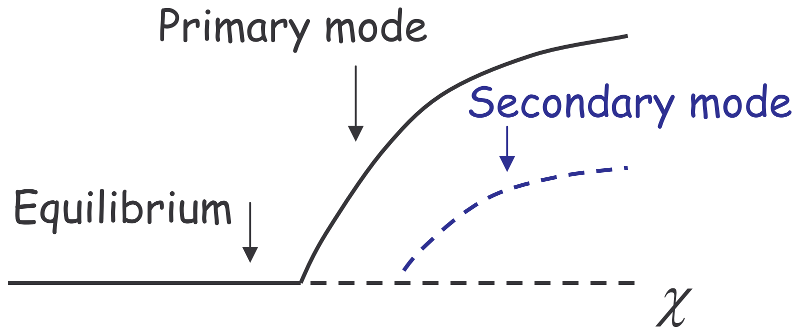
The Turing mode is not completely eliminated: Turing spots still persist in the tips of Hopf spiral waves!



Multimode localized stuctures

General idea

Driving a system out of equilibrium often induces an instability of the equilibrium state, at which some primary mode begins to grow.

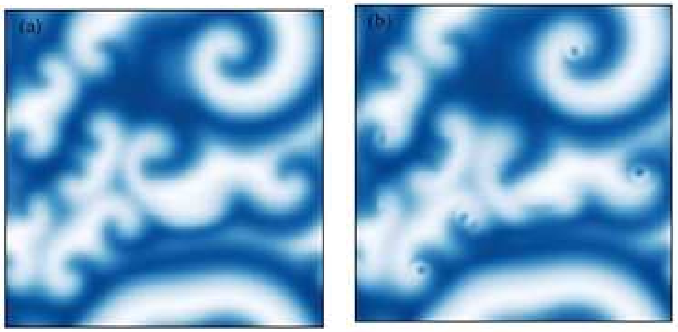


Increasing the distance from equilibrium generally induces additional instabilities of the equilibrium state, but the secondary modes that grow are often non-linearly damped by the primary mode whose amplitude is much larger.

Claim: Localized structures of a primary mode, where it's amplitude goes to zero or becomes sufficiently small, can host nonlinearly damped secondary modes, giving rise to multi-mode structures.

Demonstration with the Hopf-Turing bifurcation in the FHN model:

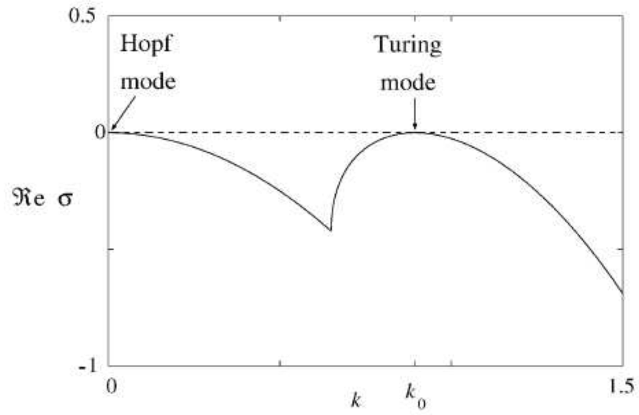
Above Hopf and slightly above Turing: →



← Above Hopf and farther above Turing

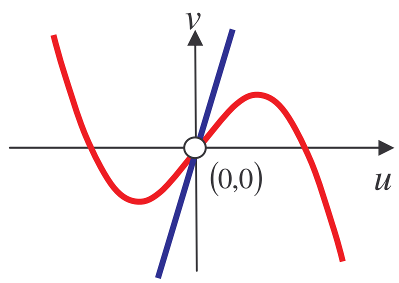
Multiple instabilities: The Hopf-Turing example

We have encountered already a Hopf-Turing bifurcation when we studied temporal forcing of oscillating systems.



Hopf-Turing bifurcations appear in many contexts, including the FHN model:

$$u_t = u - u^3 - v + \nabla^2 u$$
$$v_t = \varepsilon(u - a_1 v) + \delta\nabla^2 v \qquad a_1 < 1$$

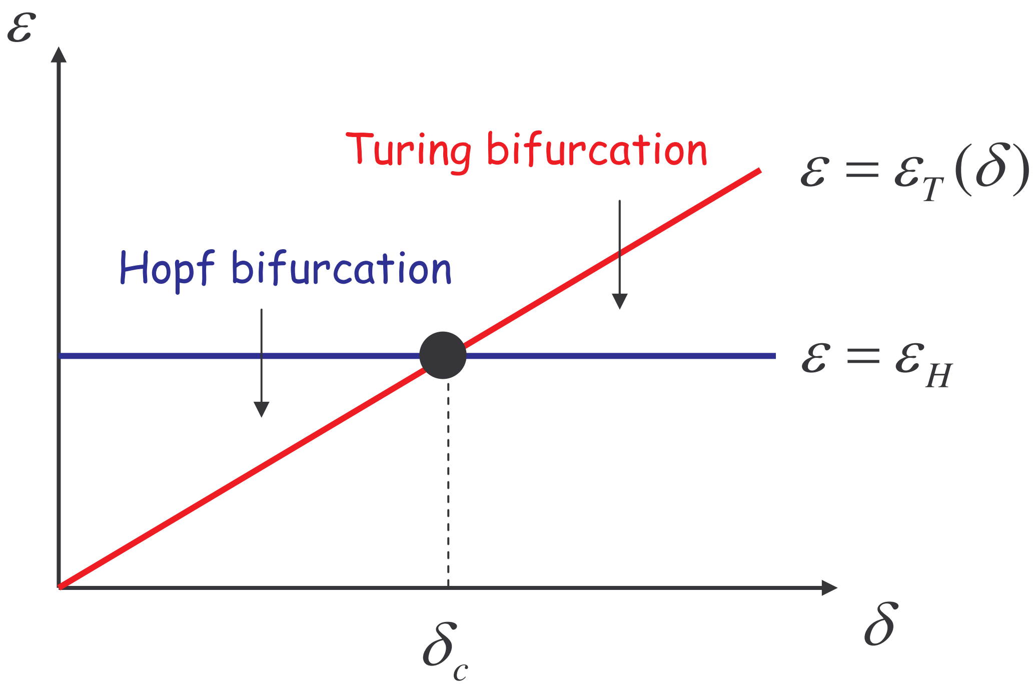


The stationary uniform state, $(u,v)=(0,0)$, loses stability to uniform oscillations as ε is decreased below $\varepsilon = \varepsilon_H = a_1^{-1}$.

For a given δ value it also goes through a Turing bifurcation as ε is decreased past

$$\varepsilon_T = \delta \Big/ \left(2 - a_1 + 2\sqrt{1 - a_1}\right)$$

Multiple instabilities: The Hopf-Turing example



A codimension-2 point at $\delta = \delta_c$ where both mode are marginally stable.

Near the codimension-2 point we can approximate solutions of the FHN model as

$$\begin{pmatrix} u \\ v \end{pmatrix} \cong \begin{pmatrix} 1 \\ c_1 \end{pmatrix} A \exp\left(ik_T x\right) + \begin{pmatrix} 1 \\ c_2 \end{pmatrix} B \exp\left(i\omega_H t\right) + c.c.$$

Amplitude of Turing mode (A)

Amplitude of Hopf mode (B)

Multiple instabilities: The Hopf-Turing example

Where the amplitudes A and B satisfy

$$A_t = \lambda A + (2k_0\partial_x - i\partial_y^2)A + (\lambda |A|^2 + \kappa |B|^2)A$$

$$B_t = (\mu + i\nu)B + \alpha\nabla^2 B + (\delta |A|^2 + \beta |B|^2)B$$

$\alpha,\ \delta,\ \beta$ are in general complex-valued constants

$\lambda = (\varepsilon_H - \varepsilon)/\varepsilon_H$ - distance from the Hopf bifurcation

$\mu = (\varepsilon_T - \varepsilon)/\varepsilon_T$ - distance from the Turing bifurcation

We can increase the variety of localized structures in the Hopf-Turing system by forcing it in time and in space. The amplitude equations then change to:

$$A_t = \lambda A + (2k_0\partial_x - i\partial_y^2)A + (\lambda |A|^2 + \kappa |B|^2)A + \gamma_k A^*$$

$$B_t = (\mu + i\nu)B + \alpha\nabla^2 B + (\delta |A|^2 + \beta |B|^2)B + \gamma_\omega B^*$$

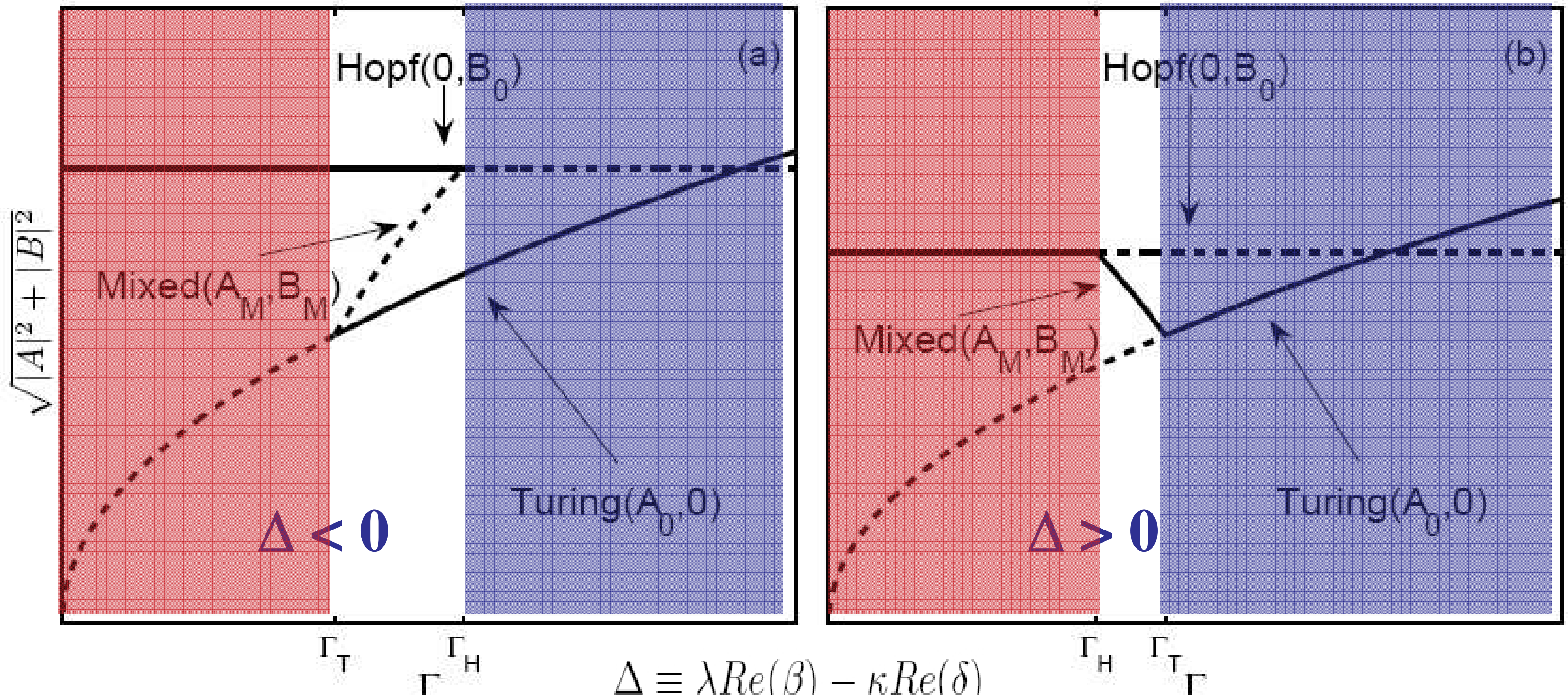
Uniform solutions of Hopf-Turing systems

Zero solution: $A=0, \; B=0$

Mixed-mode: $A=A_M, \; B=B_M$

Pure Turing mode: $A=A_0, \; B=0$

Pure Hopf mode: $A=0, \; B=B_0$



$$\Delta \equiv \lambda Re(\beta) - \kappa Re(\delta)$$
$$\Gamma = \epsilon + \gamma_k$$

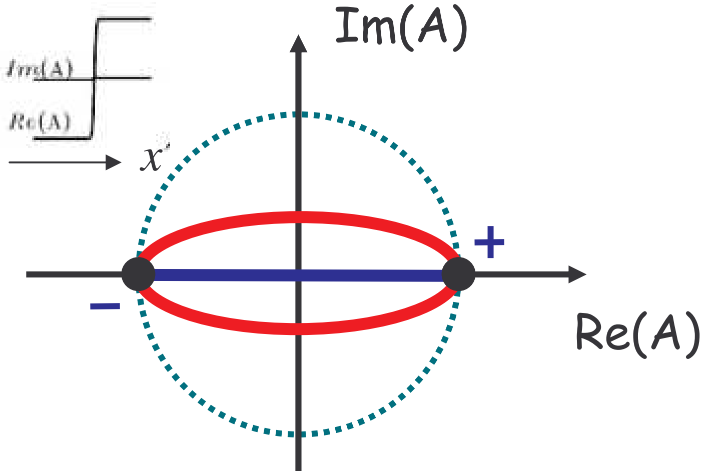
Focus on: Monostability regime of pure Hopf mode

or Monostability regime of pure Turing mode

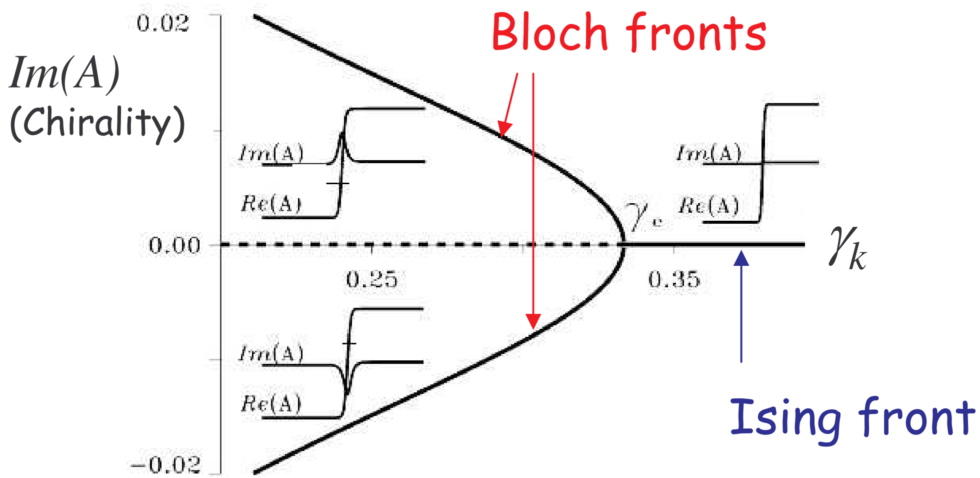
Note: in both cases $\varepsilon > 0$ and $\mu > 0$ (both modes linearly grow)

Monostability regime of pure Turing mode (B=0)

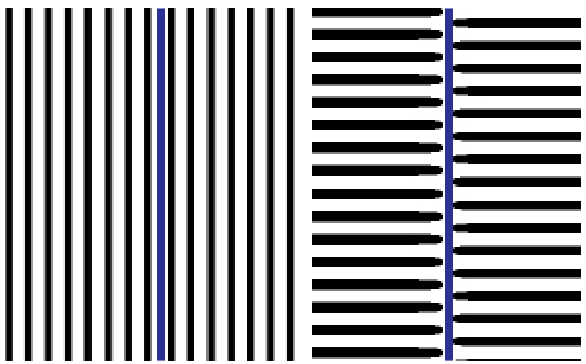
The spatial forcing fixes the phase of a Turing stripe pattern at 0 or π, creating bistability of states, fronts and vortices:



(Coullet et al. PRL 1990)

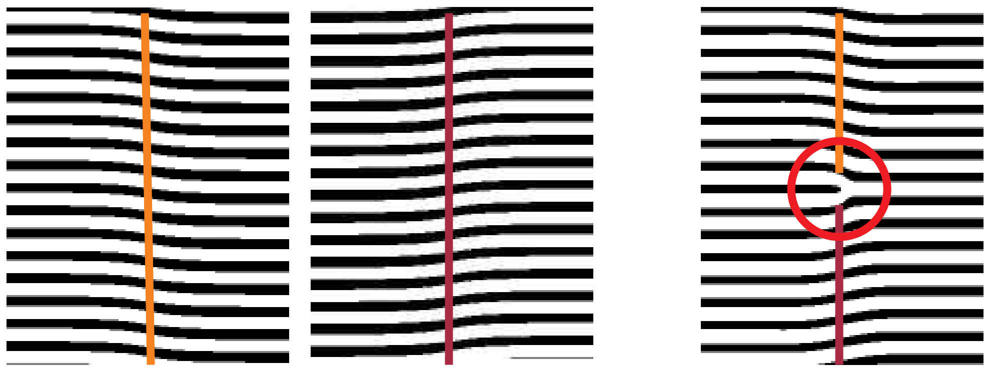


Longitudinal Ising front | Transverse Ising front



(Korzinov et al. PRL 1992)

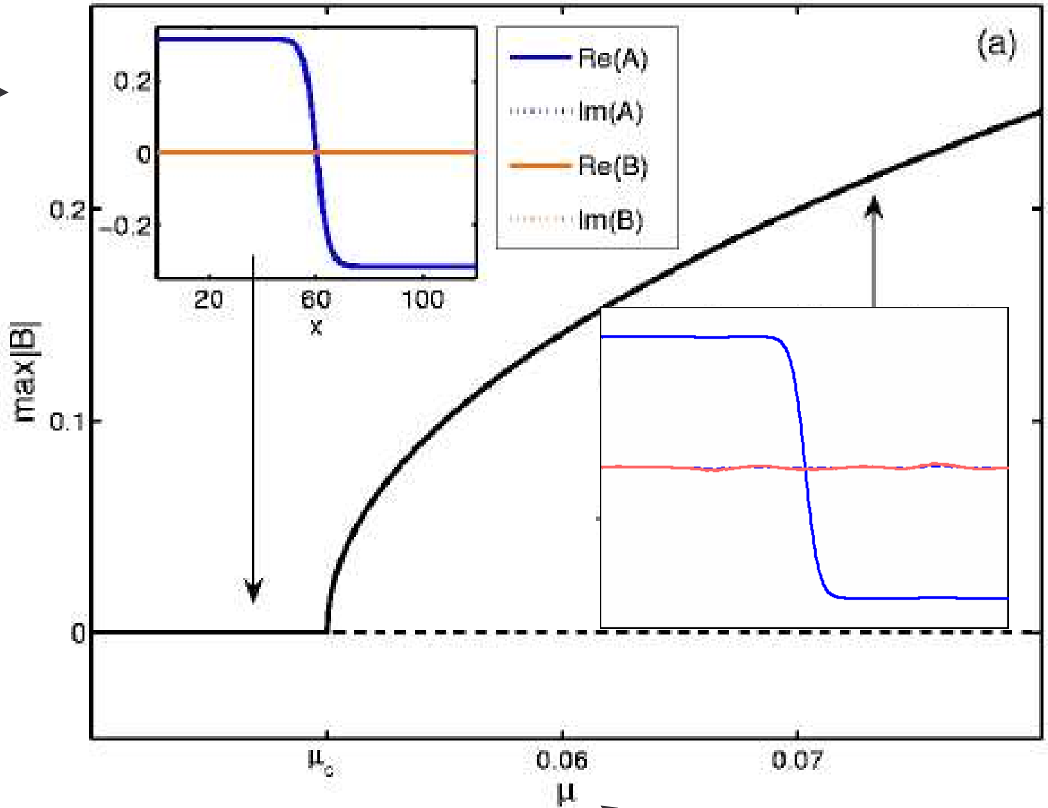
Transverse Bloch front 1 | Transverse Bloch front 2 | Bloch vortex



Localized Turing structures hosting the Hopf mode

Breathing Ising front (Lampert & Meron, EPL 2007)

Turing Ising front →



← Breathing Ising front

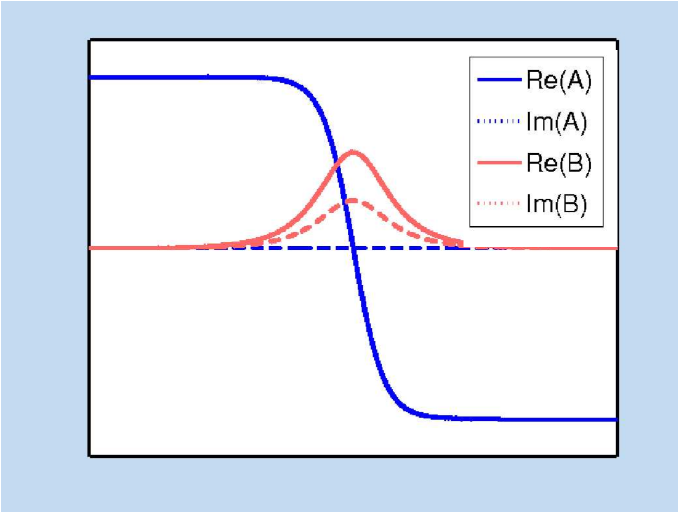
The Hopf mode does not affect the Turing mode

↖ distance from the Hopf bifurcation

The Turing Ising front hosts the Hopf mode at the front core where the Turing amplitude vanishes.

Localized Turing structures hosting the Hopf mode

Analytic solution for the breathing Ising-front



For the special case where
$\alpha = 4k_0^2$ and $\beta = \delta = \lambda = \kappa = 1$
we found the exact solution:

$$A = \sqrt{\Gamma}\, tanh[\sqrt{\Gamma - \mu}\,(x/2k_0)]$$

$$B = \sqrt{2\mu - \Gamma}\, sech[\sqrt{\Gamma - \mu}\,(x/2k_0)]e^{i\nu t + i\phi}$$

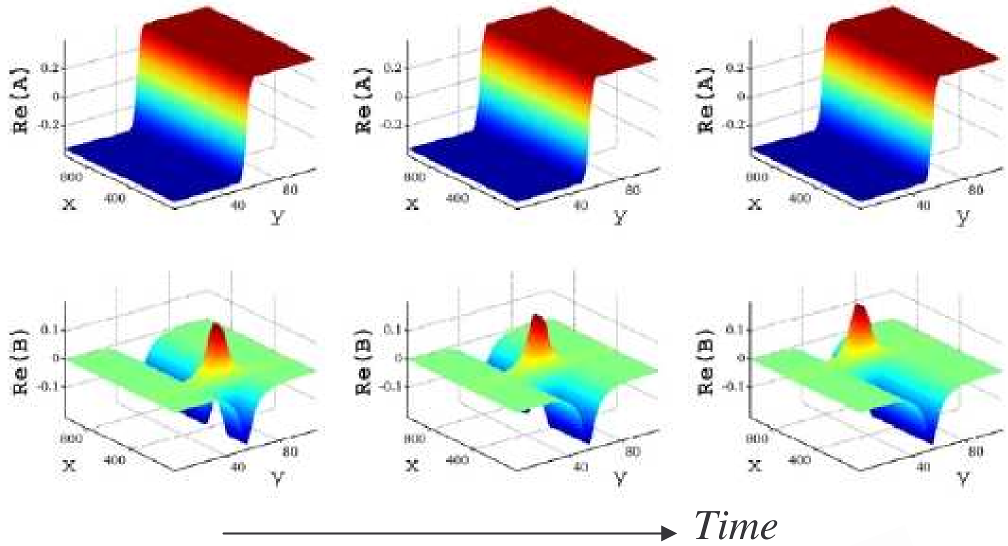
The solution appears at $\mu_c = \Gamma/2$

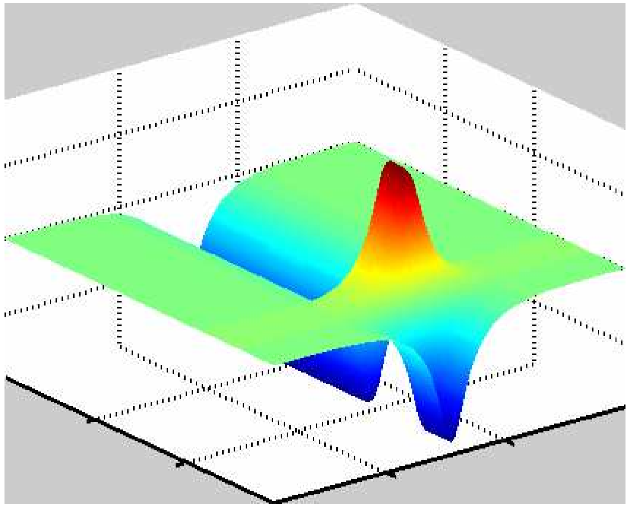
Localized Turing structures hosting the Hopf mode

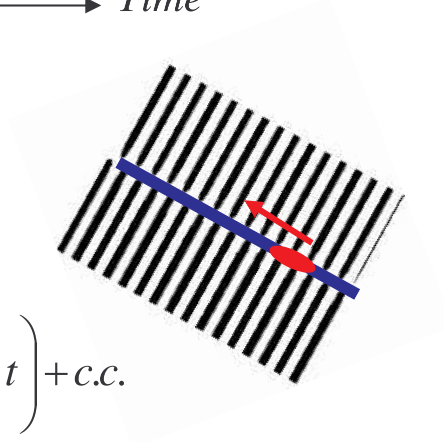
Self-organized waveguides

A breathing Ising front in 2D forms a 1D oscillatory medium along the front-core line $\Rightarrow$ traveling wave phenomena along the core line

2:1 temporal forcing leads to Hopf-pulse propagation guided by a Turing Ising front →







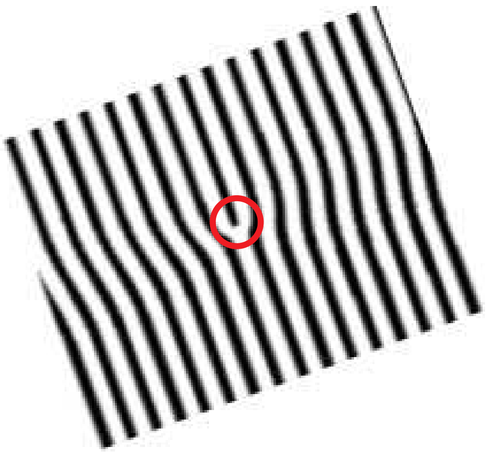
$$u(x,y,t) \cong$$

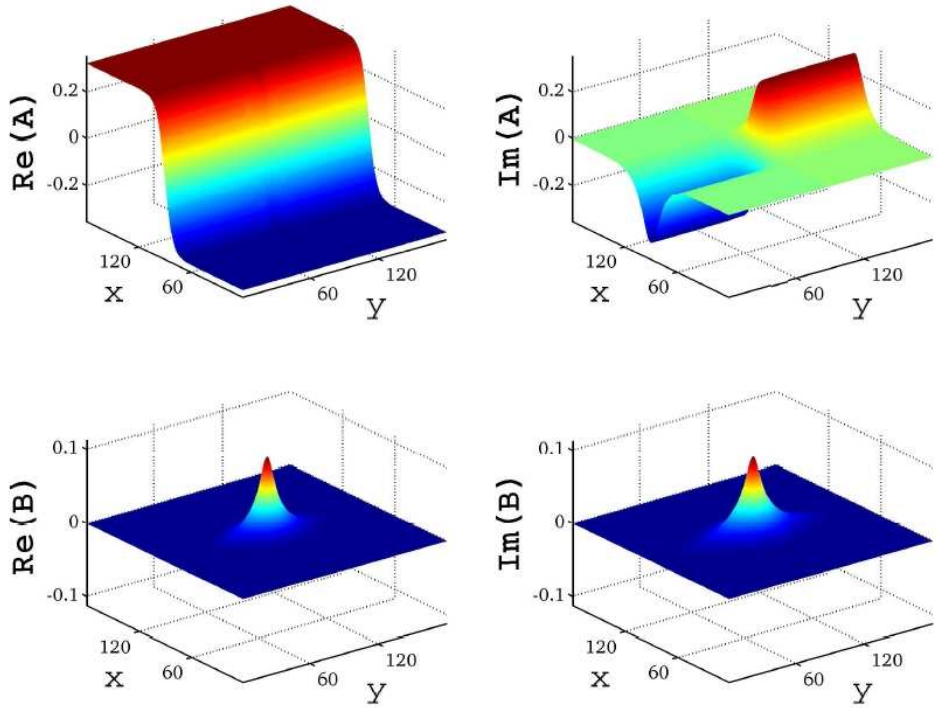
$$u_0 + c_1 A \exp\left(i\frac{k_f}{2}x\right) + c_2 B \exp\left(i\frac{\omega_f}{2}t\right) + c.c.$$

Localized Turing structures hosting the Hopf mode

Breathing Bloch vortex

A Turing Bloch vortex can host the Hopf mode at the vortex core where the Turing amplitude vanishes:



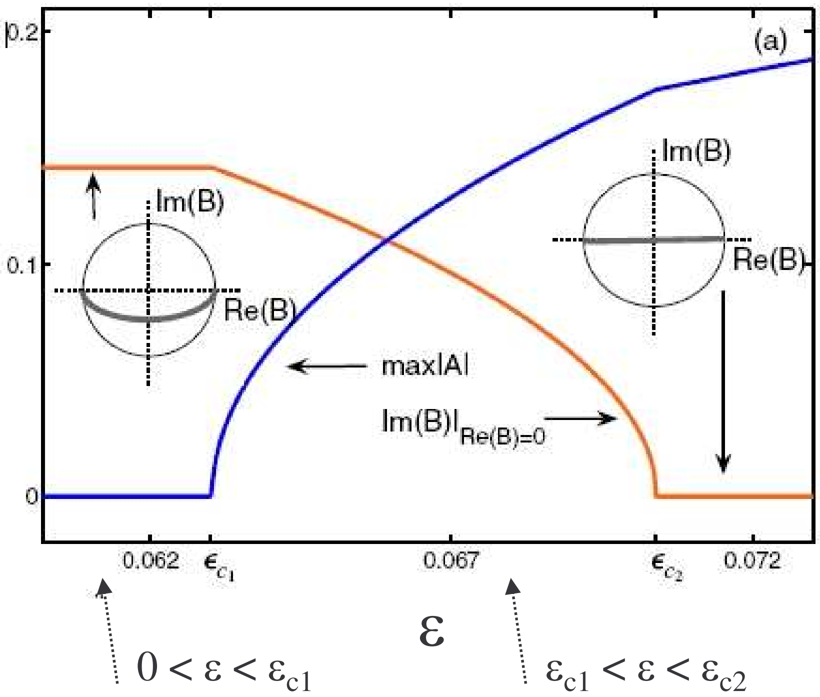


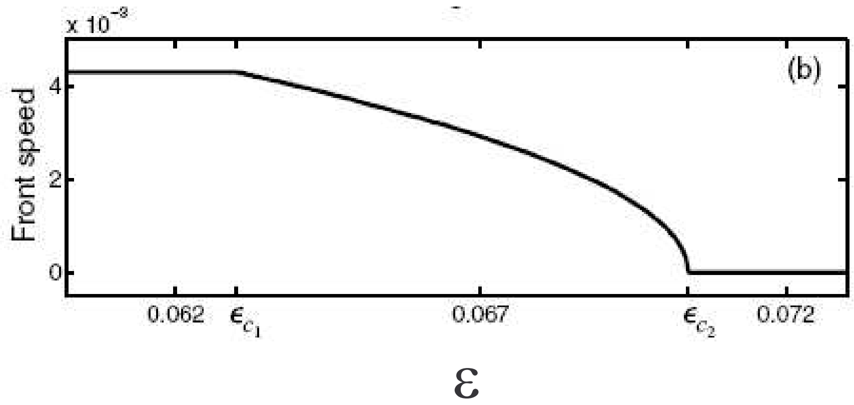
Potential application: data-storage (Coullet, Riera, Tresser, "A new approach to data storage using localized structures", Chaos 2004):

Applying a temporal forcing with $\omega_f \approx n\omega_0$, creates a breathing vortex having n different phase states.

Localized Hopf structures hosting the Turing mode

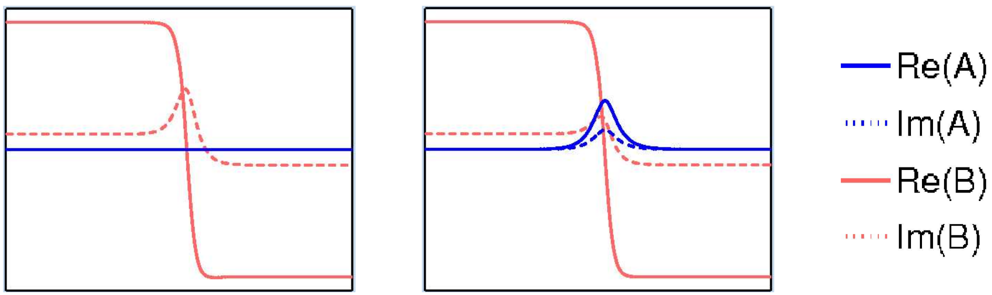
Focus on Hopf-Bloch fronts hosting the Turing mode:





Hosted Turing mode slows down Hopf-Bloch fronts ($0 < \varepsilon < \varepsilon_{c1}$) or stop their propagation ($0 < \varepsilon < \varepsilon_{c1}$).

(Lampert & Meron, EPL 2007)



Pure Hopf-Bloch front

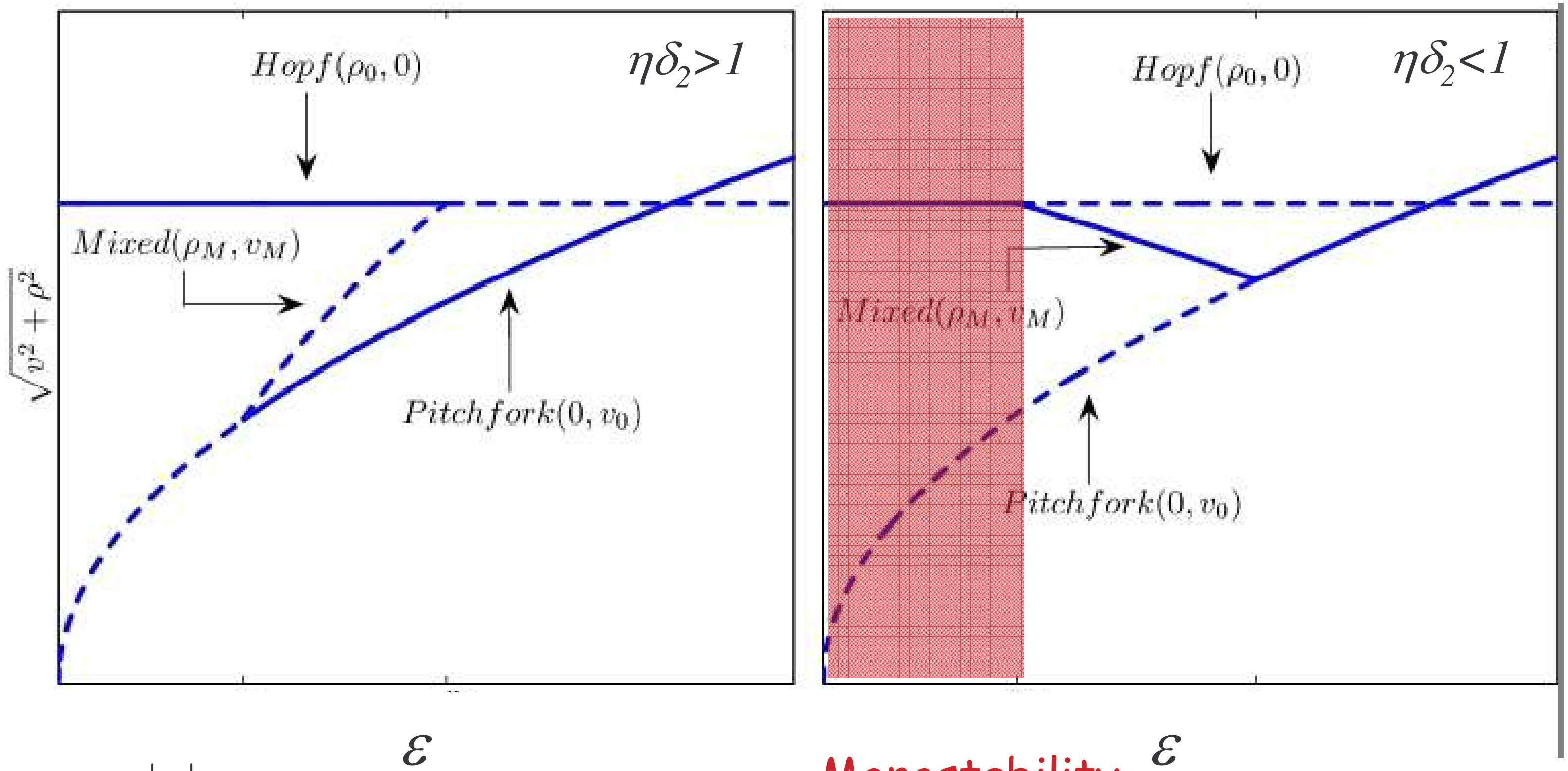
Hopf-Bloch front hosting Turing mode

Unlike the previous case, here the hosted Turing mode does affect the hosting Hopf mode.

Hosting phenomena in cusp-Hopf (pitchfork-Hopf) systems

$$A_t = (1 + i\nu)A - |A|^2 A + \nabla^2 A - i\gamma_2 A v - \delta_1 A v^2$$ } Hopf amplitude

$$v_t = \varepsilon v - v^3 + d\nabla^2 v - \eta |A|^2 v$$ Pitchfork amplitude



$\rho = |A|$

Monostability regime of pure Hopf mode

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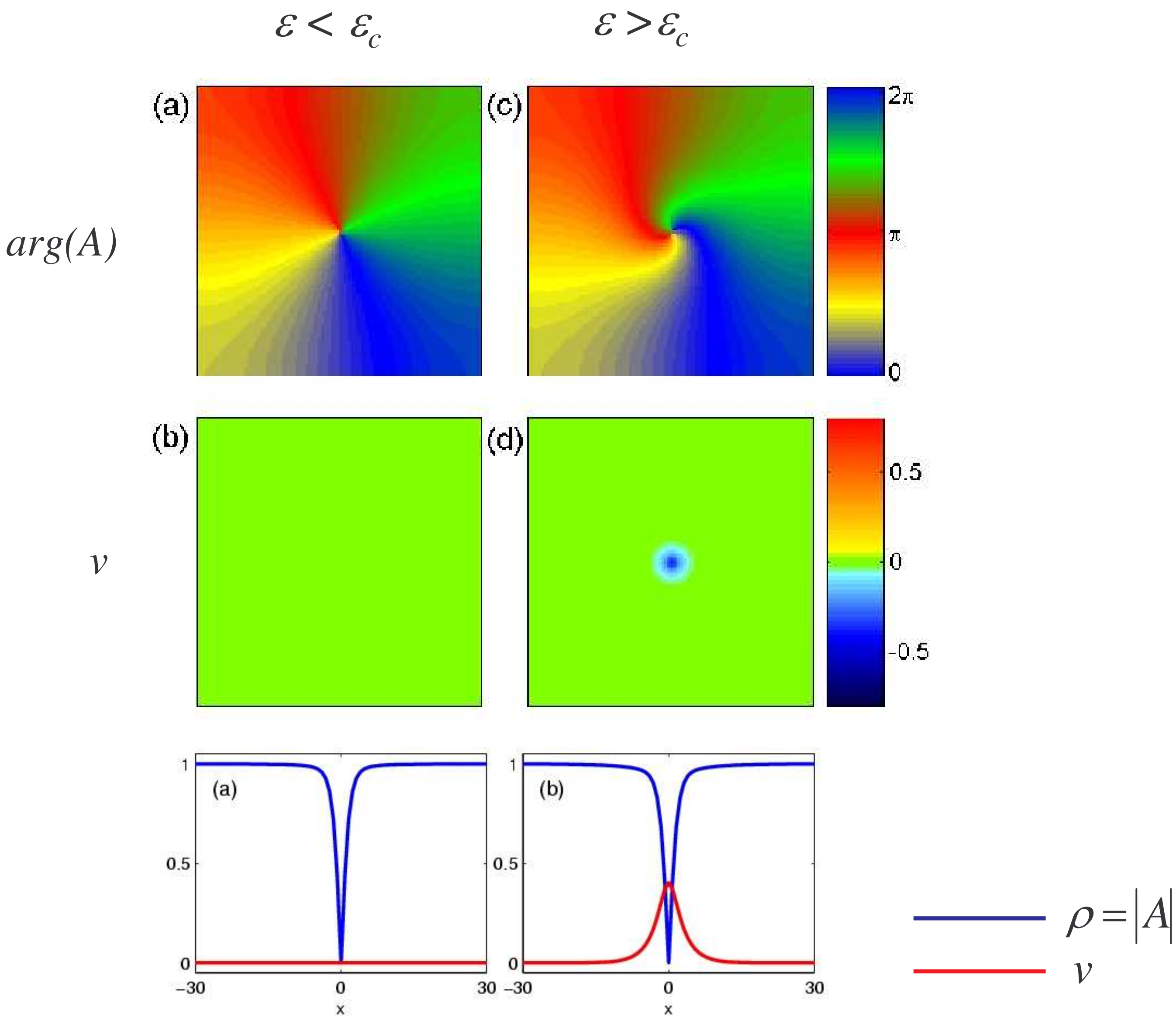
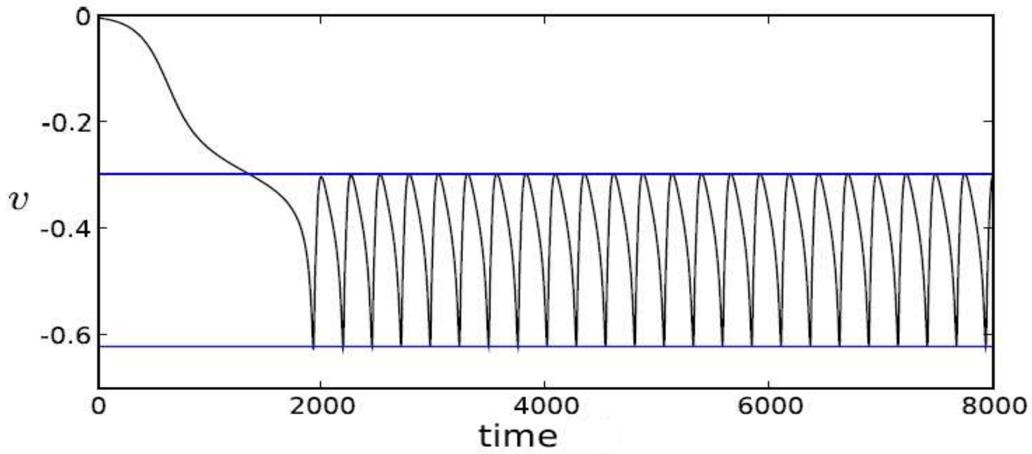
Hosting phenomena in Hopf-Pitchfork systems

A spiral-core instability: hosting the pitchfork mode



Hosting phenomena in Hopf-Pitchfork systems

A secondary spiral-core instability: successive hosting events

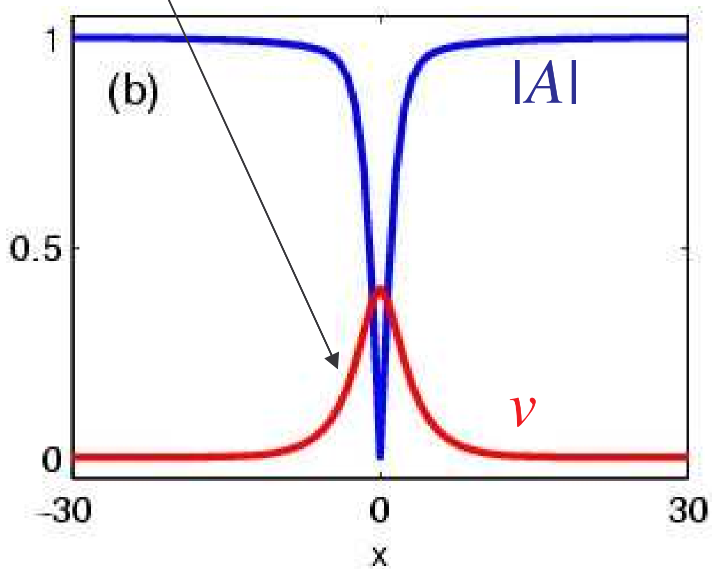


Hopf phase ϕ

Hopf amplitude $|A|$

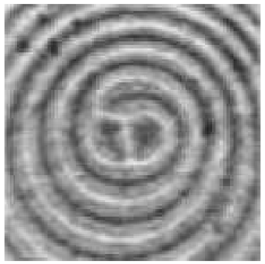
Pitchfork variable v

High $|\partial v/\partial r|$ $\Rightarrow$ High $|\partial \phi/\partial r|$ $\Rightarrow$ Small $|A|$ $\Rightarrow$ New hosting



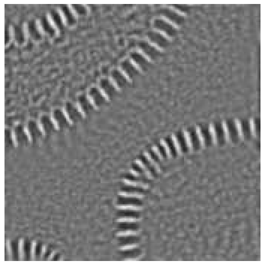
Summary

1. Higher instabilities of an already unstable zero state are important!
2. Localized structures of the first mode to grow can host other secondary modes through instabilities of pure-mode structures to mixed-mode structures. The latter can go through secondary instabilities leading to complex spatio-temporal behavior.
3. Using spatial and/or temporal periodic forcing one can control localized structures: (i) self-organized waveguides, (ii) multi-state structures.
4. The study may shed new light on earlier observations:



Rolls and hexagons in thermal convection

Assenheimer and Steinberg, 1996



Decorated kinks in vibrated granular media ?

Umbanhower et al., 1998

5. Analysis is needed!

Acknowledgement

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References

1. Lampert A. and Meron E., "Localized structures as spatial hosts for unstable modes", Europhys. Lett. 78, 14002 (1-5) (2007).