Nonlinear dynamics and pattern formation with applications to ecology

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The world we perceive with our senses is highly nonlinear:

Climate phenomena (tornados, hurricans), patterns in nature (clouds, vegetation), oscillations and synchronization (heart beats, hands clapping, stock-market dynamics), are all nonlinear phenomena.

To study phenomena of this kind we need the tools of dynamical system theory

Focus on pattern formation phenomena, their nonlinear aspects, and their relevance to the ecology of water-limited systems

General outine

Part I: Basic mechanisms of pattern formation

Basic concepts: dynamical systems, instabilities, etc.

Pattern forming systems

1. Class I – instabilities of uniform states
2. Class II – multistability of states, front dynamics

Part II: Controling patterns by temporal and spatial periodic forcing

Frequency locking, wavenumber locking in spatially extended systems

Class I and class II pattern formation mechanisms induced by the Forcing

Part III: Multimode localized structures

Localized structures, multiple instabilities, instabilities of single-mode to dual-mode localized structures

Part IV: Applications to ecology

Vegetation pattern formation in water-limited systems, plant interactions and biodiversity.

Outine of Part I: Basic mechanisms of pattern formation

Basic concepts: Open non-equilibrium systems, dynamical systems, nonlinear systems, multiplicity of states, instabilities, hysteresis

Pattern formation in nature: Symmetry breaking, universality

Canonical models of pattern forming systems: Swift-Hohenberg, FitzHugh-Nagumo

Pattern forming systems class I: Instabilities of uniform states
Examples, linear stability analysis, nonlinear analysis – amplitude equations, secondary instabilities

Pattern forming systems class II: Multiplicity of stable states
Examples, fronts, front instabilities, labyrinthine patterns, spiral waves, spiral turbulence.

Basic concepts: open non-equilibrium systems

The natural systems we mentioned represent open non-equilibrium systems, that continuously exchange materials and energy with their environments.

One of the first tasks in exploring such systems is defining what constitutes the "system" and what constitutes the system's "environment".

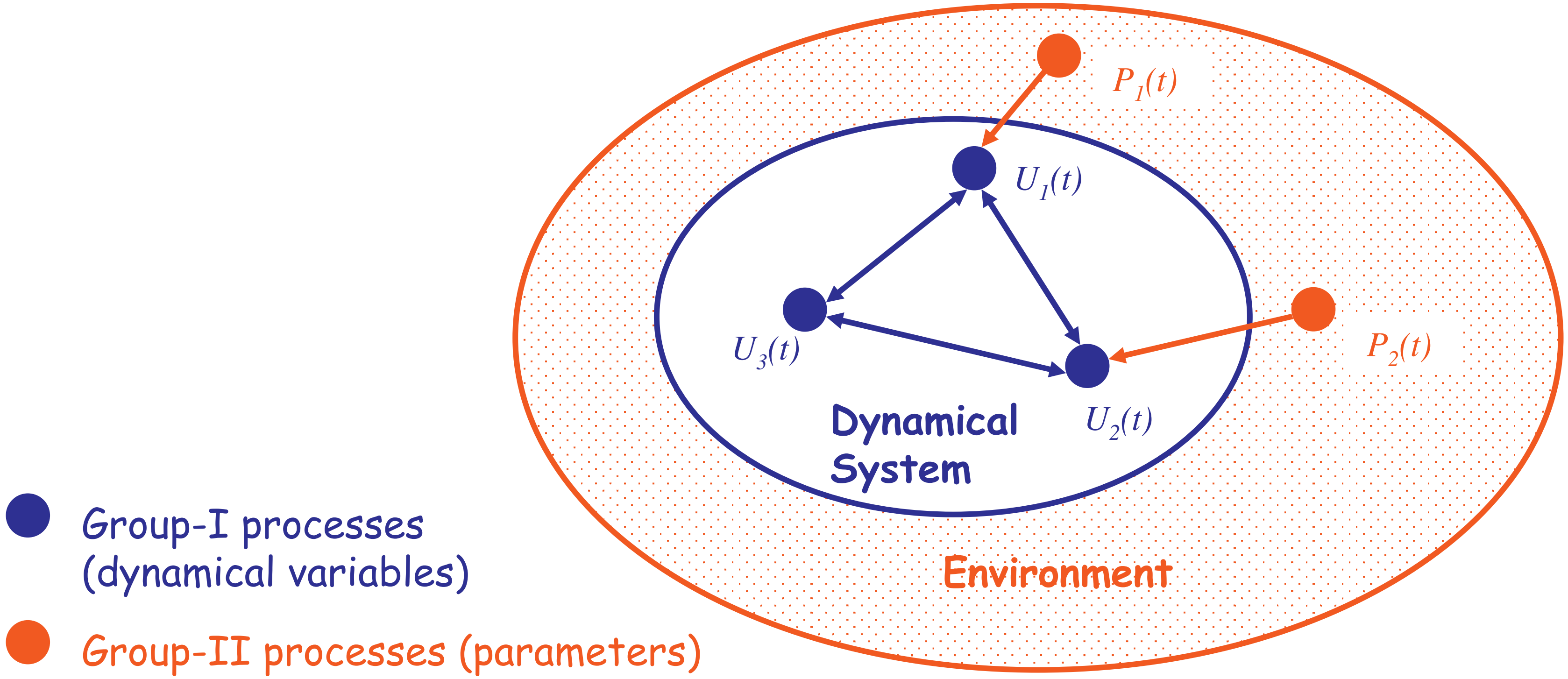
The distinction between the system and its environment depends on the processes they involve, which can be divided into two general groups:

1. Group I: Processes that mutually affect one another through various feedbacks.
2. Group II: Processes that affect Group-I processes but are hardly affected by them.

Using this process classification we can introduce the concept of a dynamical system.

Basic concepts: defining a dynamical system

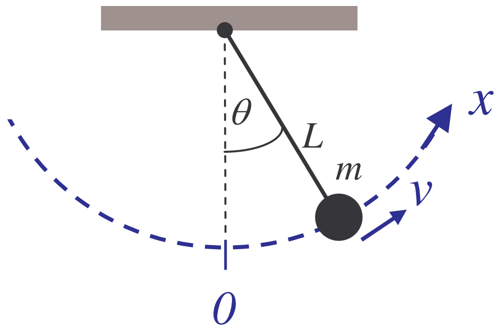
Dynamical system - a set of group-I processes, quantified by dynamical variables, $U_i(t)$, that evolve in time according to some laws of motion. These laws describe the mutual relationships among the dynamical variables and the manners by which they are affected by parameters, $P_i(t)$, representing group-II processes, which define the system's environment.



Basic concepts: linear and nonlinear dynamical system

Example of a dynamical system: A pendulum with a friction

The system is described by Newton's 2nd law which we can express in terms of the bead position, $x(t)=L\theta(t)$, and the bead velocity $v(t)=dx/dt$:



$$\frac{dx}{dt}=v \qquad \frac{dv}{dt}=-\frac{b}{m}v-\frac{g}{L}\sin x$$

where the bead is subjected to the air-friction force $F=-bv$ and we assumed small oscillation amplitudes.

Group-I processes (system):

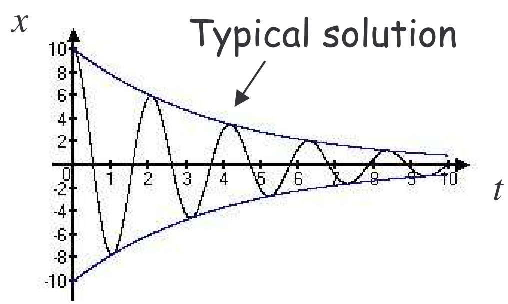
Bead-position (angle), $\boldsymbol{x}$.
Bead-velocity, $\boldsymbol{v}$.

Group-II processes (environment):

Friction with air, $\boldsymbol{b}$.
Gravitational force, $\boldsymbol{mg}$.

This is a linear dynamical system (assumption of small amplitude oscillations).

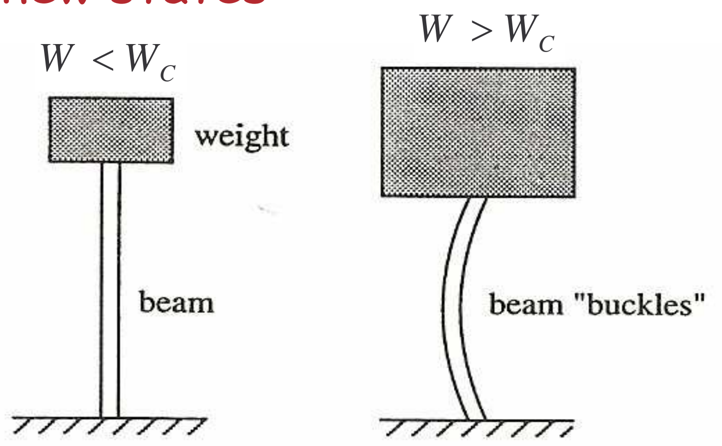
Dynamical systems representing natural phenomena are nonlinear in general.



Nonlinearity can induce instabilities to new states

The buckling of a beam:

There is a critical weight, W_C, beyond which the beam buckles.



Let,

u - measure the degree of buckling

$\lambda = (W - W_C)/W_C$ - the deviation from the critical point

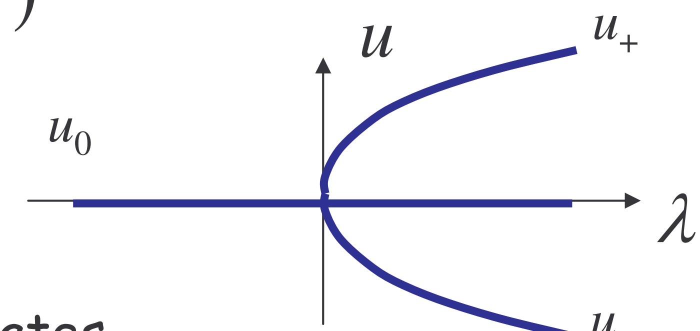
Close to the critical point the system can be described by the nonlinear dynamical system

$$\dot{u} = \lambda u - u^3 = f(u) \qquad (\dot{u} \equiv du/dt)$$

Steady state solutions $(\dot{u} = 0)$:

$u_0 = 0$ - the unbuckled state

$u_{\pm} = \pm\sqrt{\lambda}$ - two symmetric buckled states

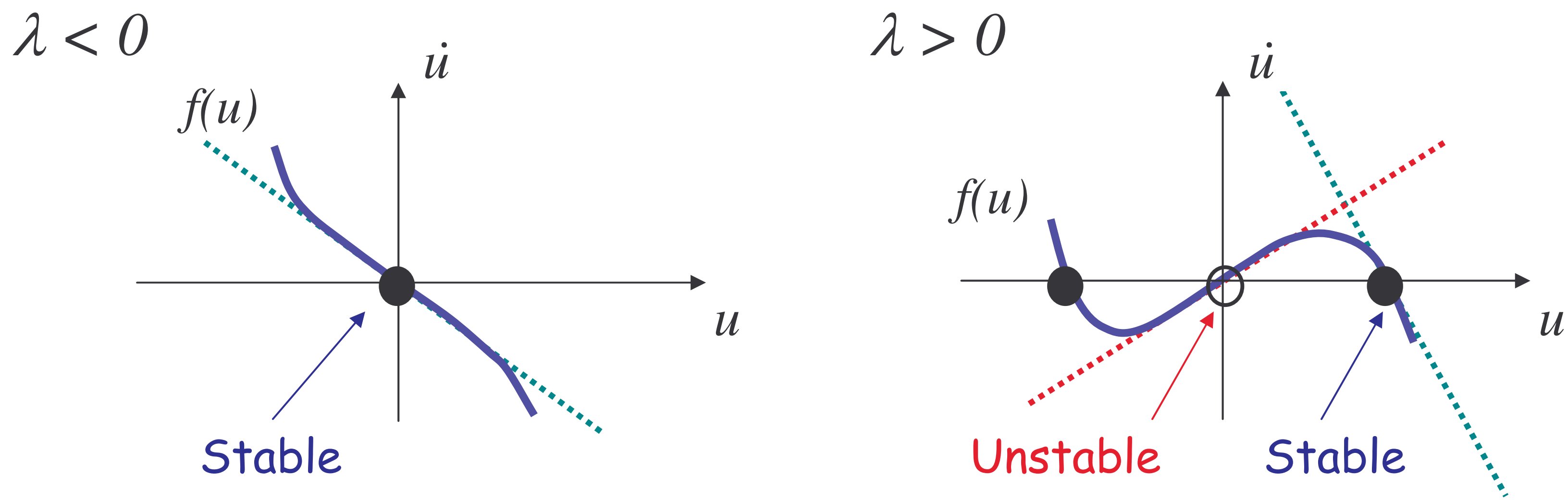


Basic concepts: linear stability

The steady state solutions tell us that beyond the critical weight, or when $\lambda>0$, the buckled states appear, but they do not tell us that the system will necessarily evolve towards these states. This type of information is contained in the stability properties of the various states.

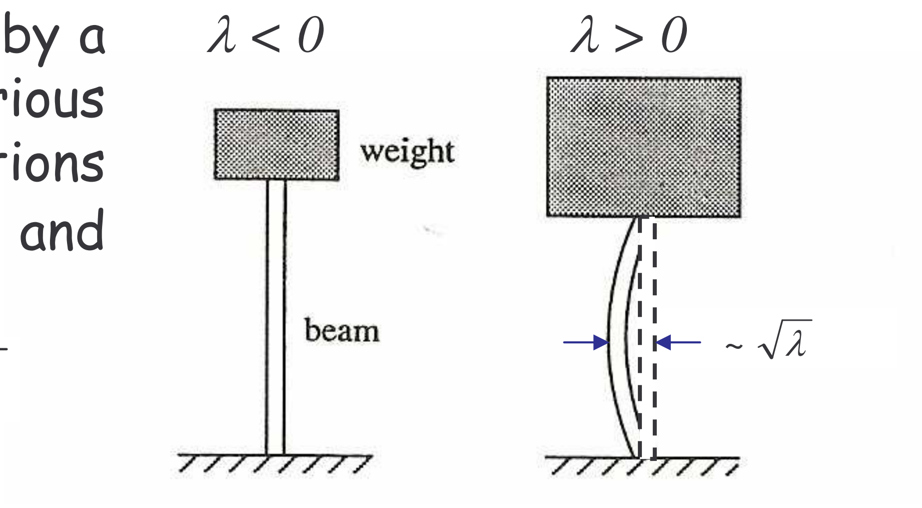
Linear stability: A solution (state) u_s is stable if any small perturbation of u_s decays in the course of time.

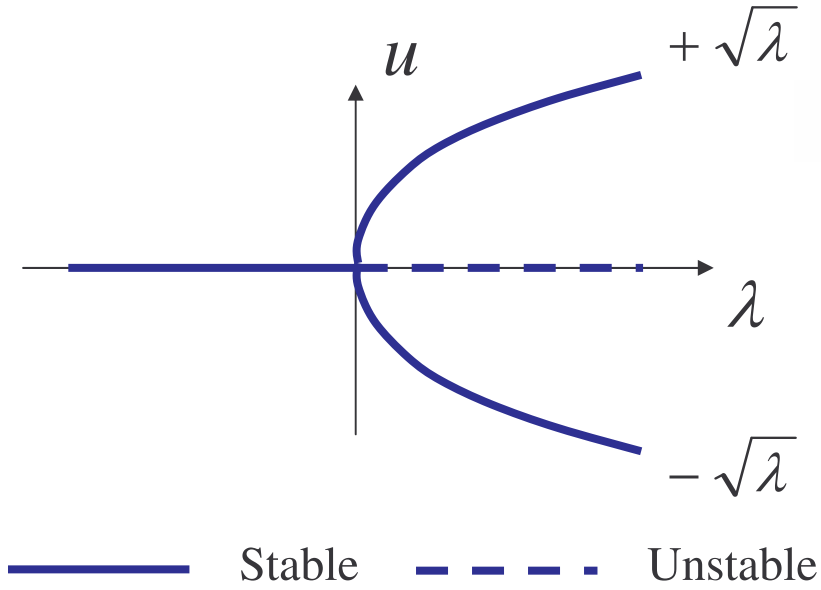
A graphical view: A solution u_s is stable (unstable) if the slope $f'(u_s)$ of $\dot{u} = f(u) = \lambda u - u^3$ at u_s is negative (positive)



Basic concepts: bifurcation diagrams

We describe such an instability by a diagram that shows the various steady-state solutions as functions of the control parameter λ, and their stability properties:





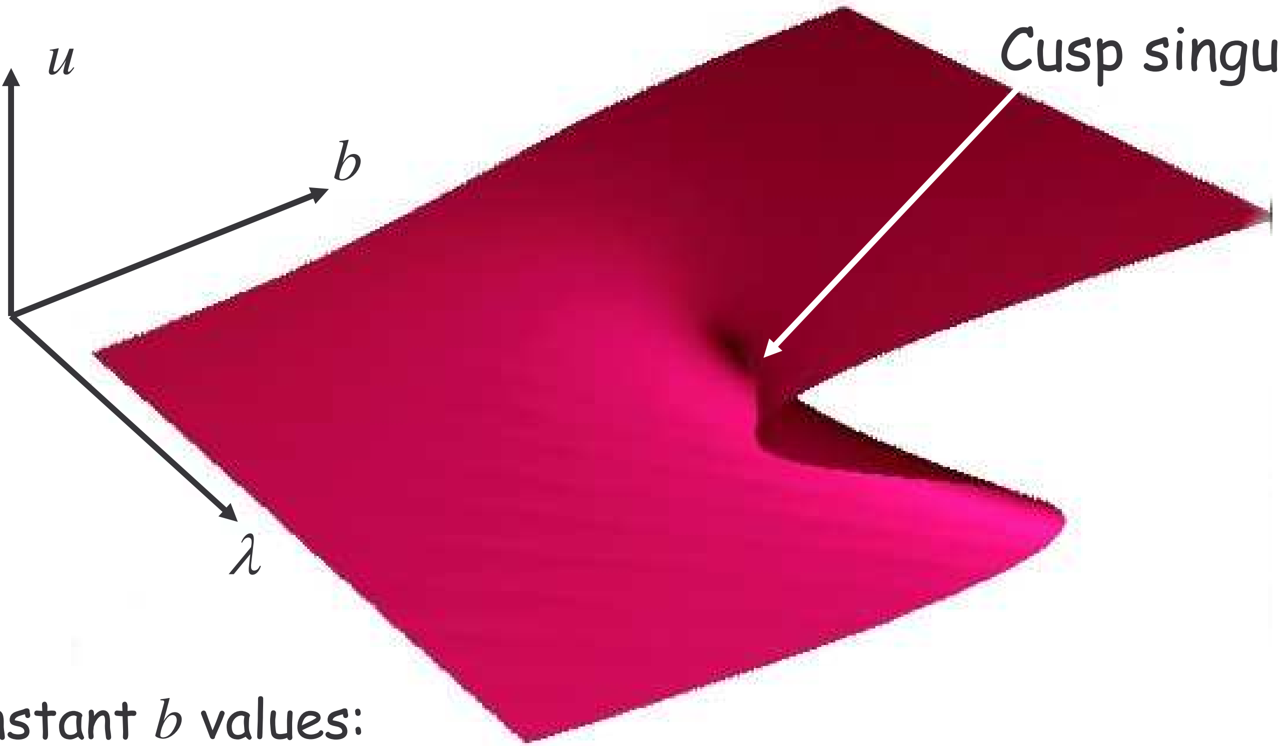
Diagrams of this kind are called bifurcation diagrams

This particular instability is called the pitchfork bifurcation

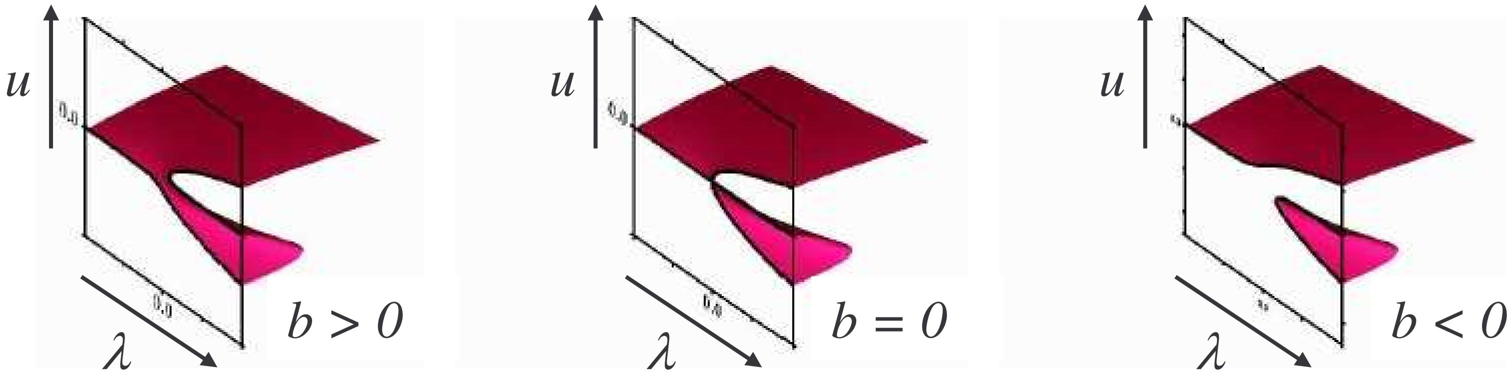
In practice the two buckled states need not be perfectly symmetric, and the instability may favor one state over the other. Mathematically, such imperfections are described by additional terms that break the symmetry $u \to -u$

Basic concepts: the cusp singularity

$$\dot{u} = \lambda u - u^3 + \cancel{au}^2 + b \quad = 0$$



Cuts at constant b values:



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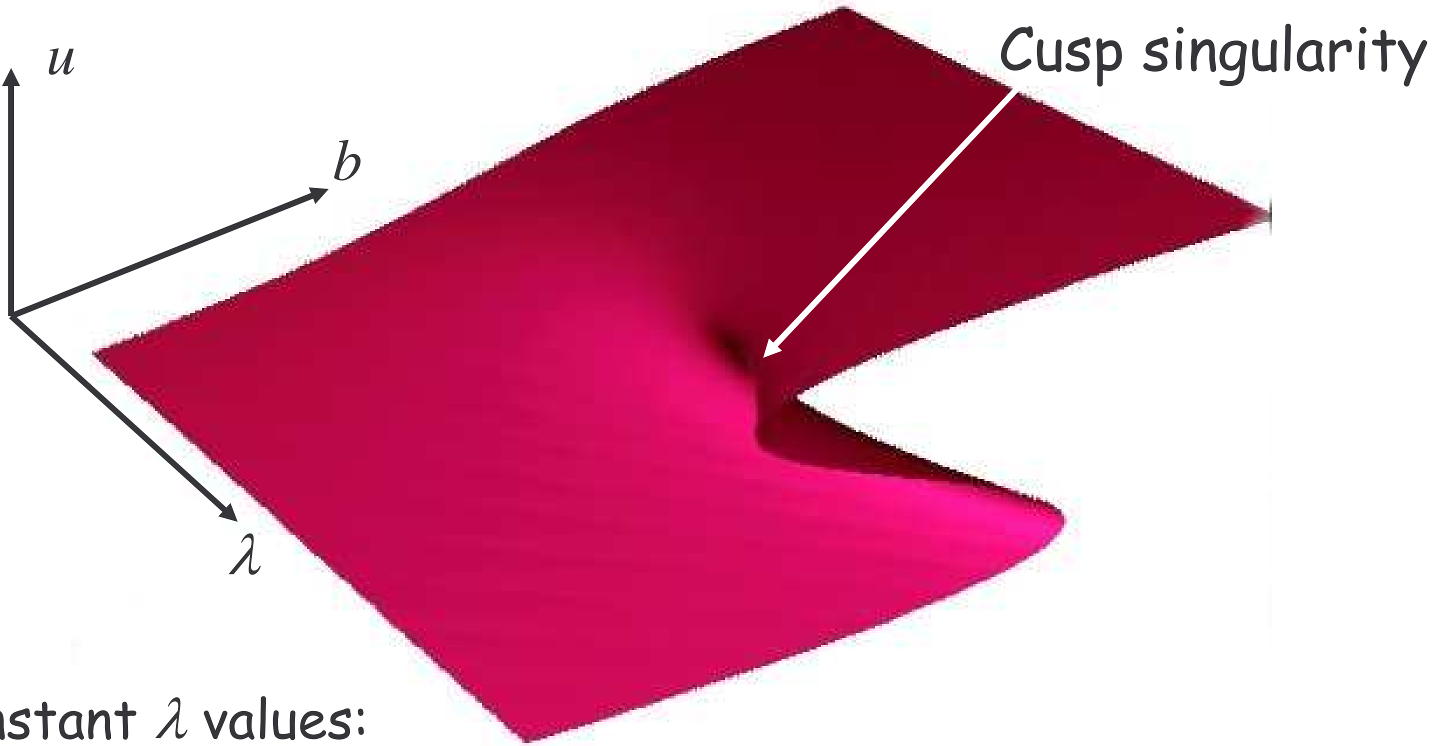
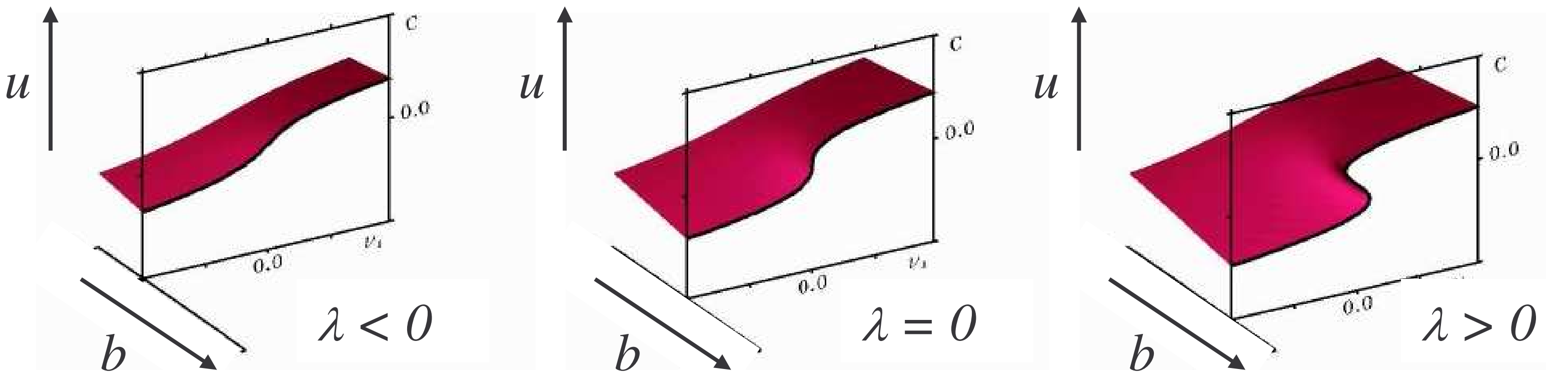
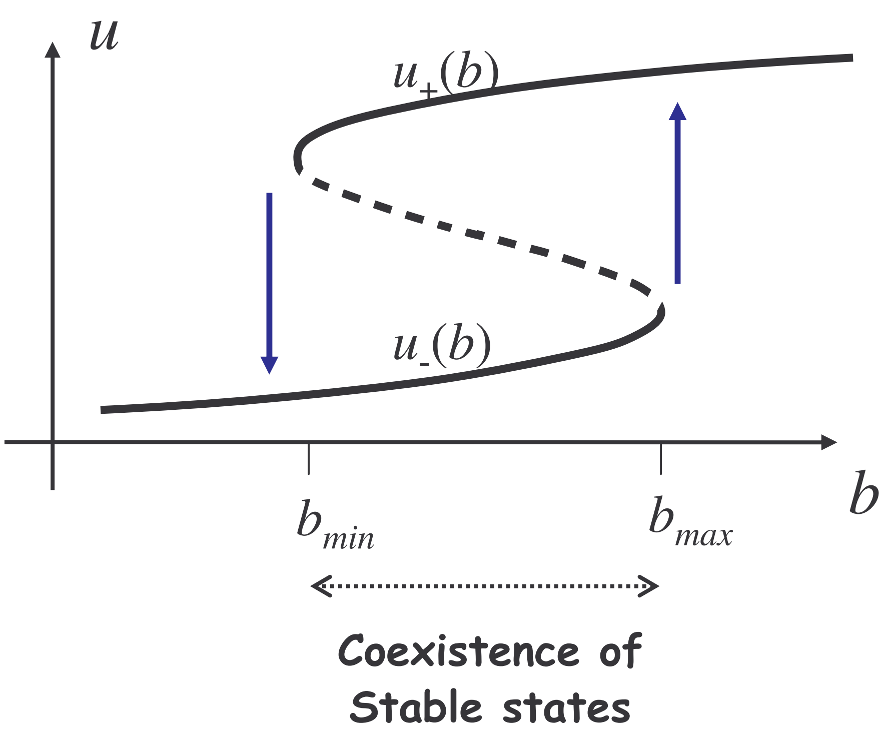


Cuts at constant λ values:



Basic concepts: hysteresis



Hysteresis phenomena:

Different transition points upon increasing and decreasing b.

Introduced in the context of magnetism, exists in numerous other contexts

In ecology - catastrophic shifts (**Scheffer et al. Nature 2001**):

Desertification – sudden decrease in biological productivity as a result of climate fluctuations (droughts) and human- related activities (over-grazing).

Sudden loss of transparency and vegetation in shallow lakes subjected to human-induced eutrophication.

Hysteresis in Escher's art

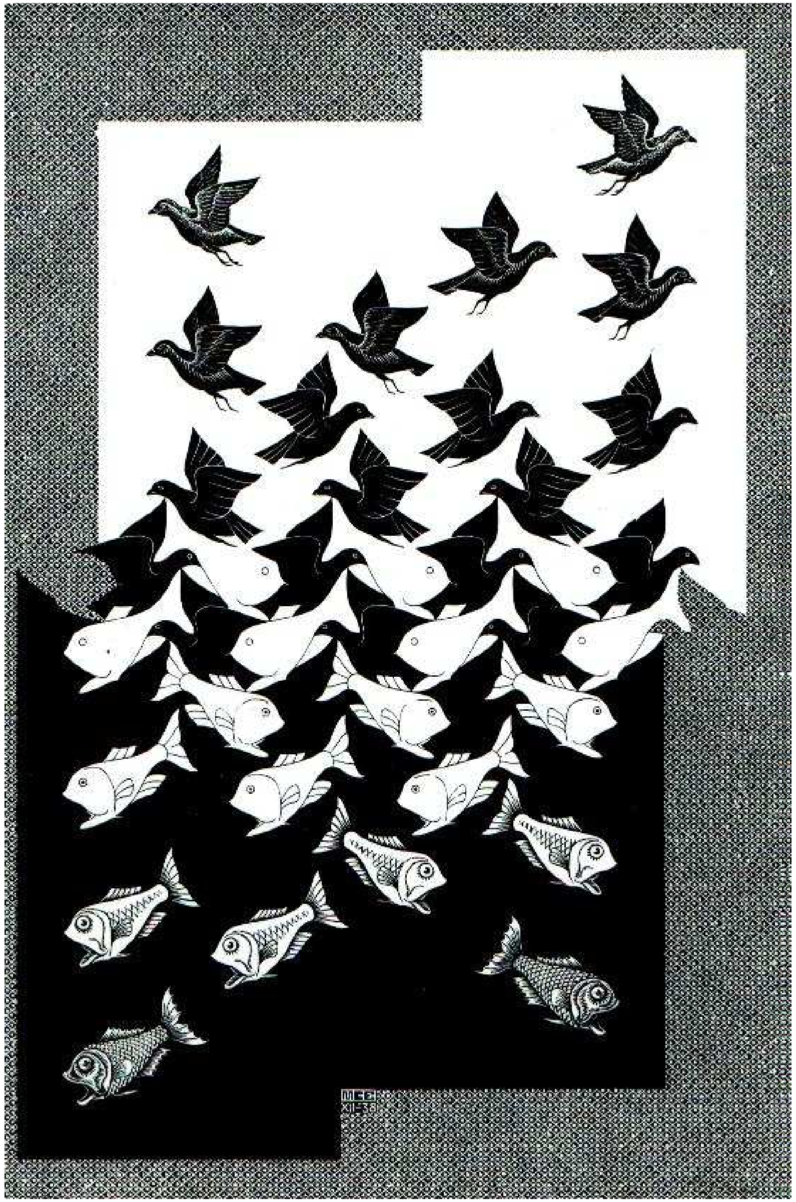
Birds

Coexistence of birds & fish

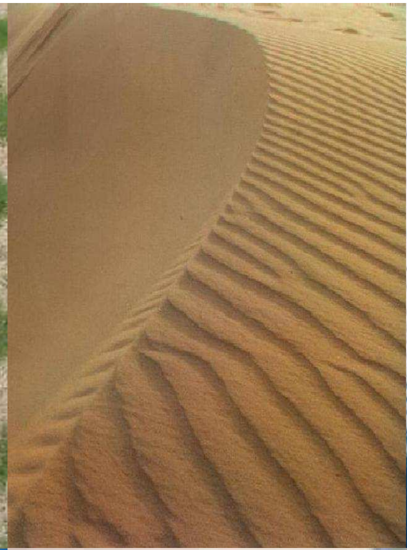
Fish to birds

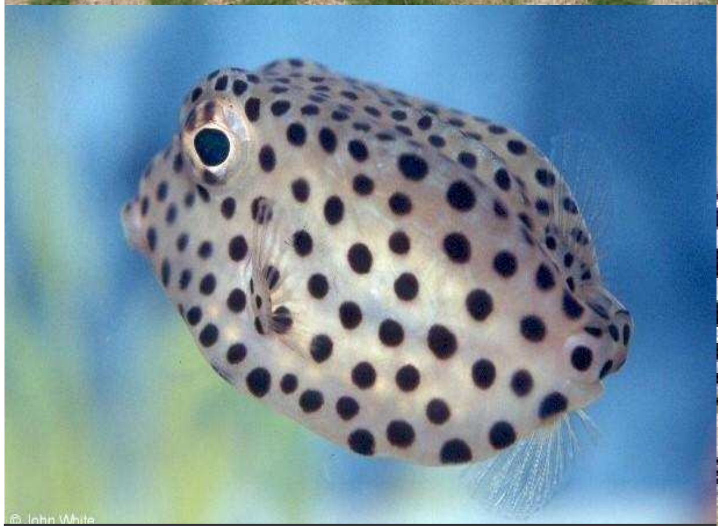
Birds to fish

Fish



Pattern formation in nature





- *Symmetry breaking*
- *Universality*
- *Emergence*

Ben Gurion University, Ehud Meron - www.bgu.ac.il/~ehud

Canonical models of pattern forming systems

Numerous experimental and theoretical studies support the interpretation of natural patterns as symmetry breaking phenomena.

Of these, studies based on mathematical models have been very instrumental in understanding the mechanisms underlying these phenomena.

In some contexts the model equations are based on firm theories and reproduce observed behaviors with quantitative accuracy. Examples are the Navier-Stokes equations of fluid dynamics, and the Maxwell equations in nonlinear optics.

In most other contexts, especially when the system's complexity increases, mathematical models are less accurate. However, they can still be very helpful in capturing the qualitative behavior of a system, and in understanding the specific mechanisms responsible for it.

Along with models that are geared to reproduce and predict behaviors of specific systems, simplified models, focusing on general aspects of pattern formation rather than on the details of a specific system, have appeared. A few of them have become canonical models.

Canonical models of pattern forming systems

The Swift-Hohenberg model

$$u_t = \lambda u - u^3 - \left(\nabla^2 + k_c^2\right)^2 u$$

The FitzHugh-Nagumo model

$$u_t = \lambda u - u^3 - v + \nabla^2 u$$
$$v_t = \varepsilon (u - a_1 v - a_0) + \delta \nabla^2 v$$

$$u_t = \frac{\partial u}{\partial t}, \qquad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \qquad \begin{array}{l} -\infty < x < \infty \\ -\infty < y < \infty \end{array}$$

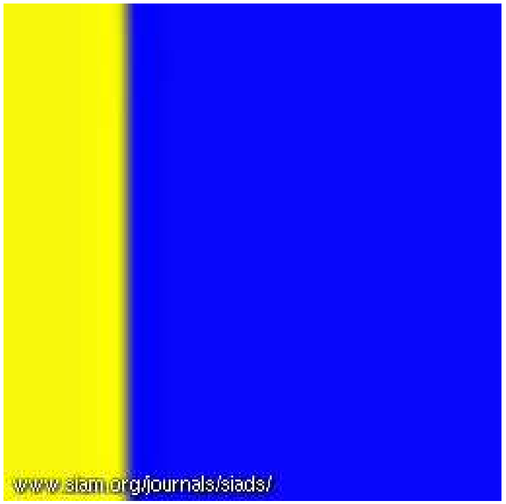
u(x,y,0)
v(x,y,0) ⟶ ■ ⟶ ***u(x,y,t)***
v(x,y,t)

Classes of pattern forming systems

Models of this kind help us understand basic mechanisms of pattern formation and classify pattern forming systems accordingly.

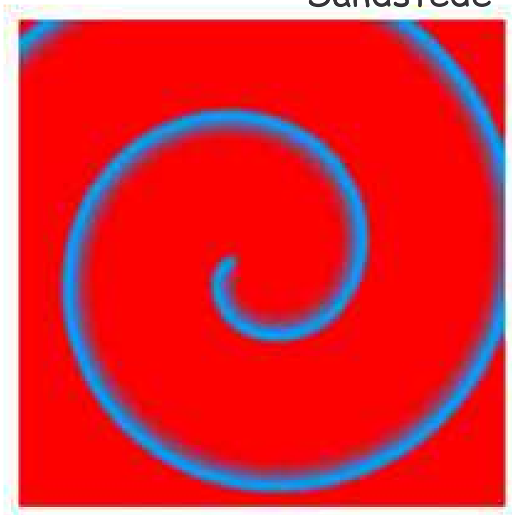
We will distinguish here among two classes which we illustrate with the following model simulations:

Class I: Instabilities of uniform states



Class II: Multistable systems

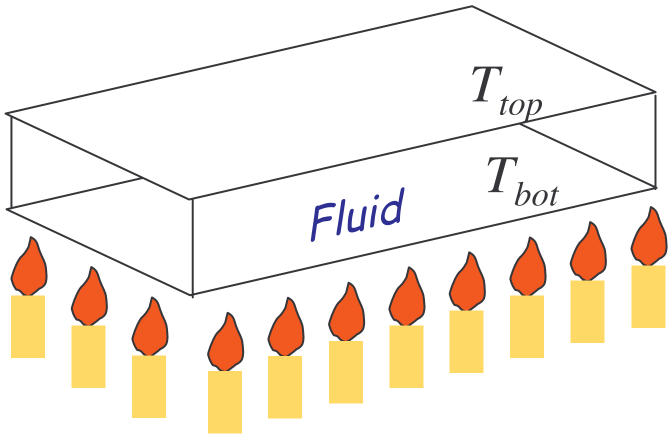
Sandstede



Class III: Excitable systems

Class I: Instabilities of uniform states

Thermal convection (Rayleigh-Bénard): heating a fluid layer from below

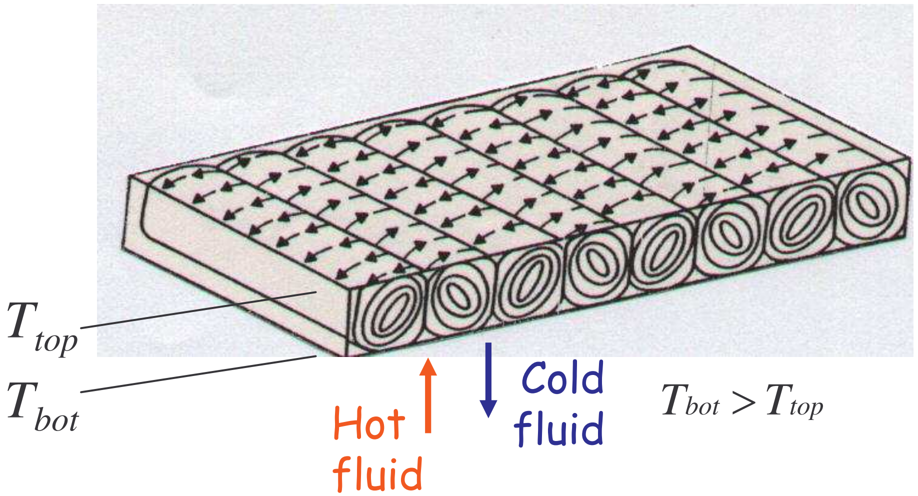


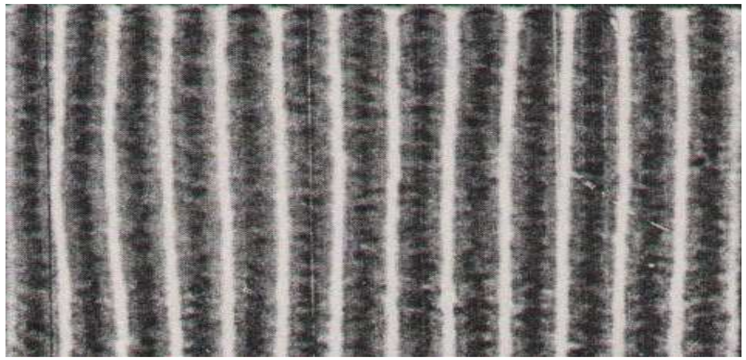
Below a critical temperature difference,

$$\Delta T = T_{bottom} - T_{top} < \Delta T_C$$

the fluid is at rest – heat is transferred by molecular conduction

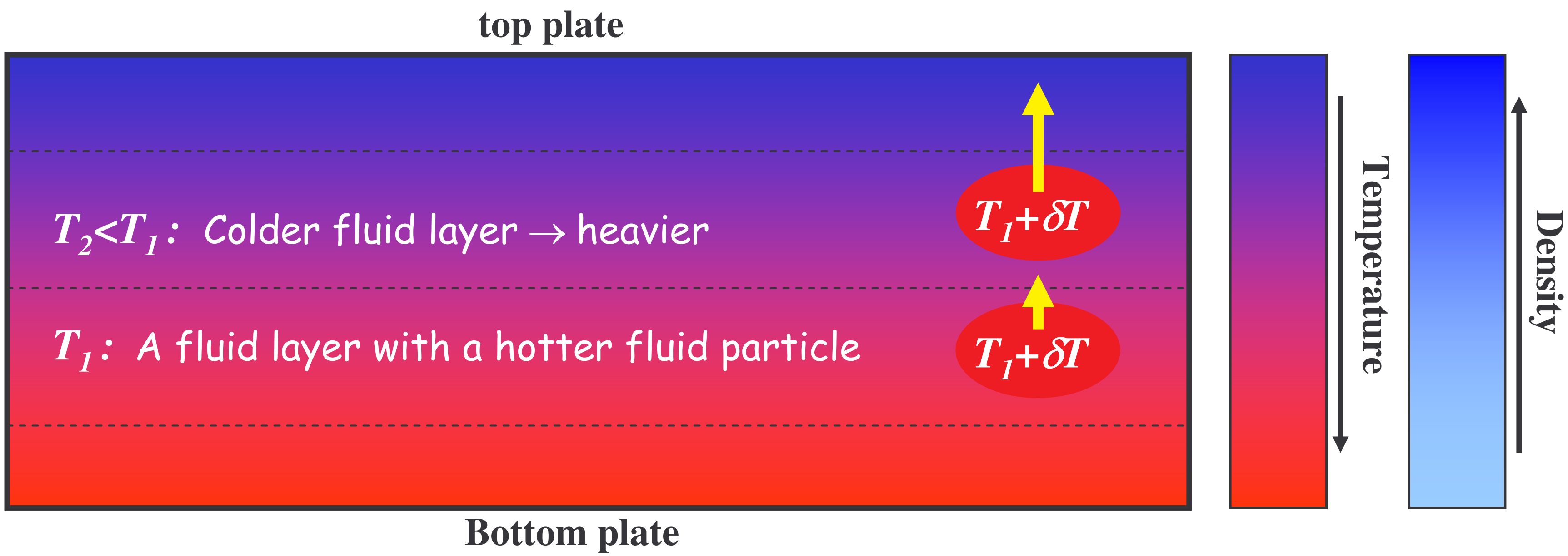
Above ΔT_C convection sets in the form of ordered parallel roles





Visualization by illuminating from below (index of refraction changes with temperature periodically).

Why do we have an instability when the temperature difference is large enough?



Imagine a fluctuation in which a fluid particle at some height has a temperature which is slightly higher than the surrounding fluid at that height. Because of thermal expansion the fluid particle will have a density lower than that of the surrounding fluid (i.e. will be lighter) and will tend to move upward. As it moves upward the surrounding fluid becomes yet colder and the buoyancy force upwards increases.

Why does not the instability occur when the temperature difference is not large enough?

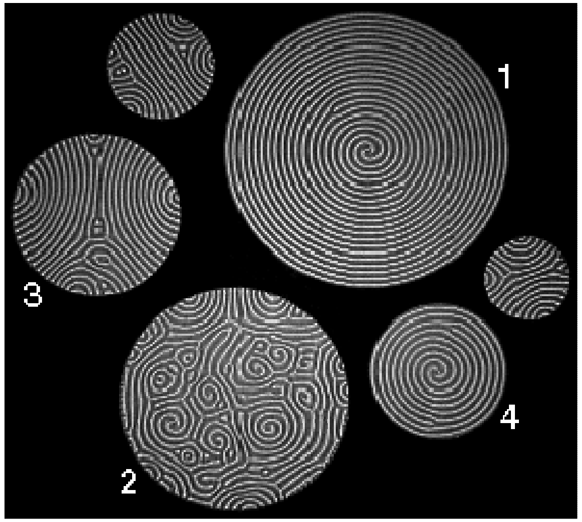
There are processes that stabilize the rest state:

Fluid viscosity: induces transfer of linear momentum from the up-moving fluid particle to its neighborhood, thus reducing its momentum and speed.

Thermal conduction: induces diffusion of heat from the fluid particle to its colder neighborhood, thus reducing the buoyancy force that drives the fluid particle upward.

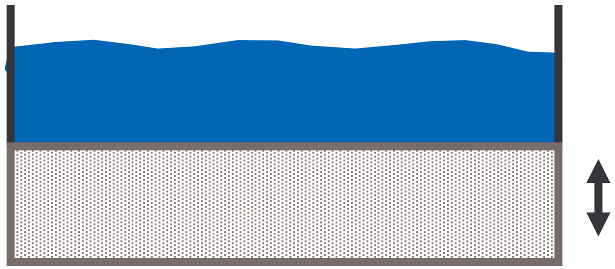
$\Rightarrow$ There should be a critical temperature difference, ΔT_c, at which the stabilizing factors just balance the destabilizing buoyancy force.

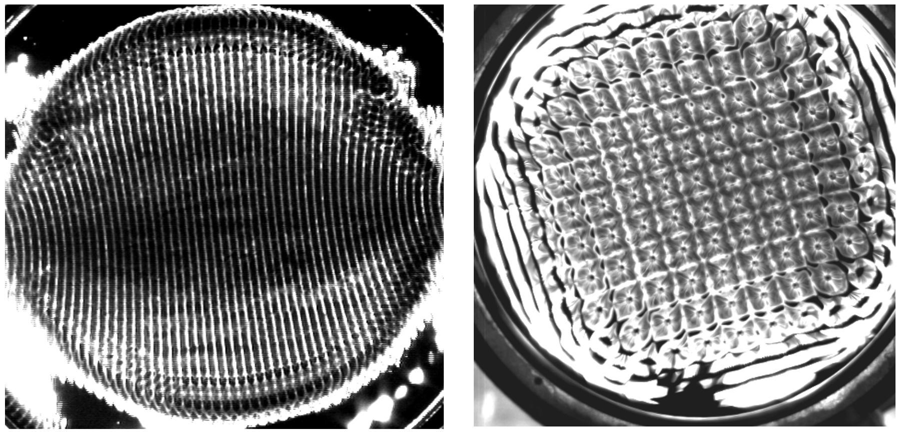
Bodenschatz lab. 1997: small Prandtl number fluid experiments



Class I: Instabilities of uniform states

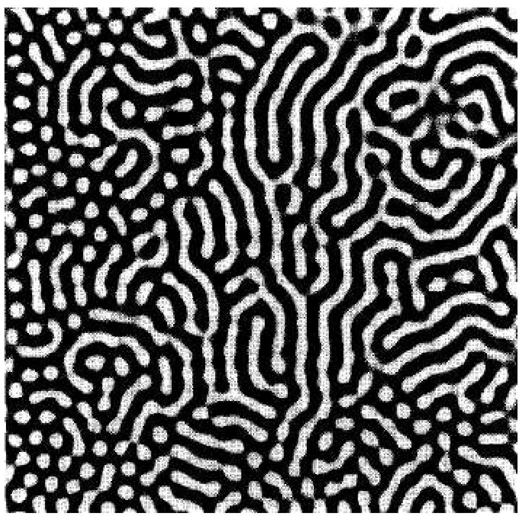
Farady surface waves
(stripes and squares)



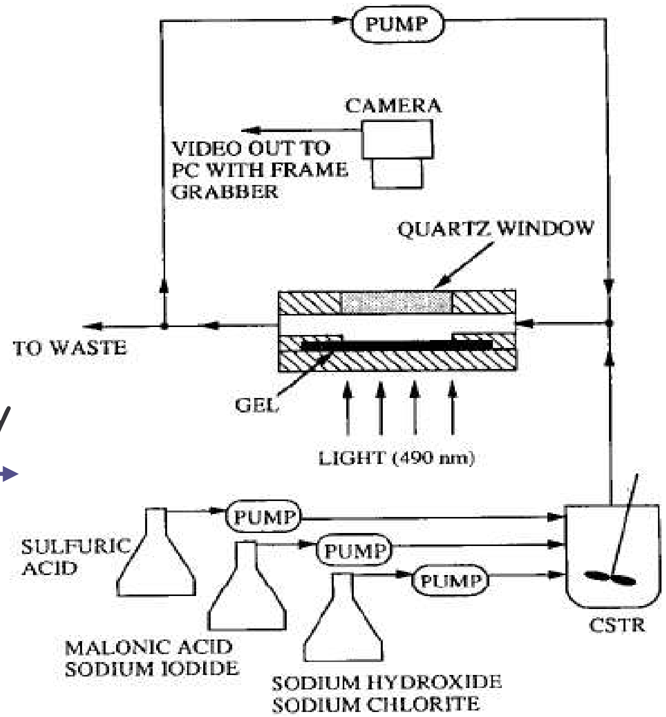


Stephen W. Morris

Chemical reactions



CIMA reaction:
Turing patterns
Vigil, Ouyang and Swinney



Class I: Linear stability analysis of the uniform state

A powerful method that provides information about the instability threshold of a uniform state and the nature of the mode that grows at the instability point is linear stability analysis. Let's illustrate this method using the Swift-Hohenberg (SH) model:

$$\frac{\partial u}{\partial t} = \lambda u - u^3 - \left(\frac{\partial^2}{\partial x^2} + k_0^2\right)^2 u \qquad -\infty < x < \infty$$

Consider the stationary uniform solution $u_s=0$. This solution is linearly stable if any infinitesimally small perturbation decays in the course of time. It is linearly unstable if there exists a perturbation that grows in time.

Writing $u(x,t) = u_s + \delta u(x,t)$, where $\delta u(x,t)$ is an infinitesimally small perturbation, we obtain after linearizing in $\delta u(x,t)$:

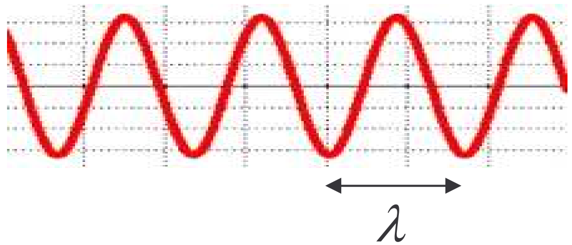
$$\frac{\partial \delta u}{\partial t} = \lambda \delta u - \left(\frac{\partial^2}{\partial x^2} + k_0^2\right)^2 \delta u$$

Class I: Linear stability analysis of the uniform state

Since any perturbation can be represented as a Fourier integral and since the equation for δu is linear, it is sufficient to study the stability of $u_s=0$ to the growth of any Fourier mode,

$$\delta u(x,t) = A(t)e^{ikx} + c.c. \quad |A| \to 0 \qquad \left(e^{ikx} = \cos kx + i\sin kx \right)$$

In other words we study the growth or the decay of sinusoidal perturbations with wavenumber k or wavelength $\lambda=2\pi/k$:



$$k = \frac{2\pi}{\lambda}$$

The amplitude of the perturbation satisfies the equation

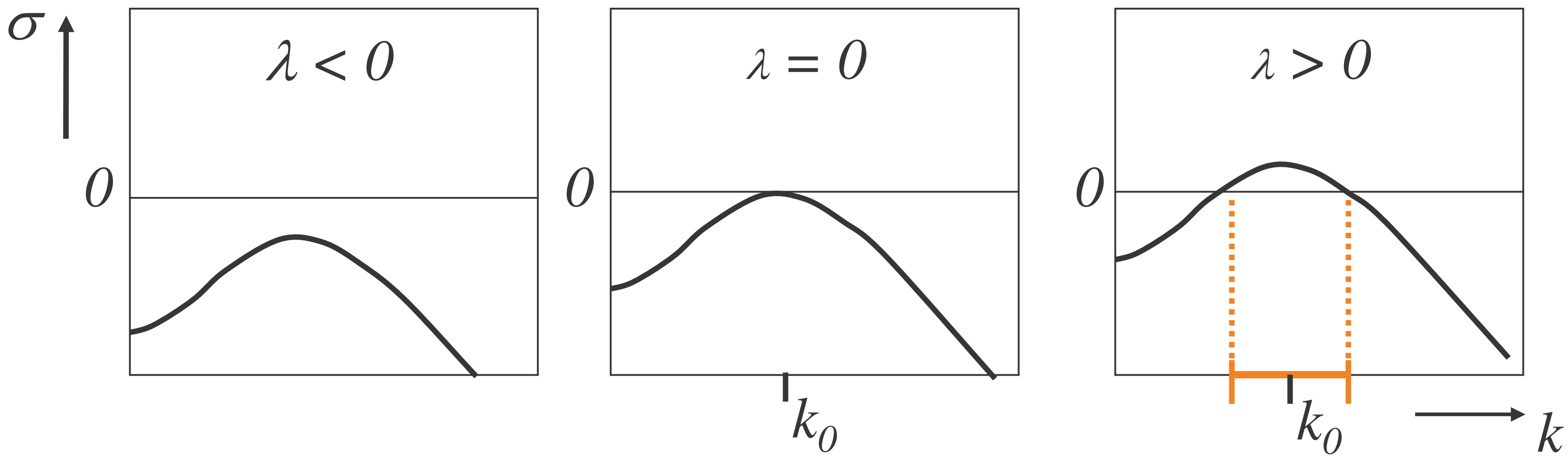
$$\dot{A} = \sigma(k)A, \qquad \sigma = \lambda - \left(k_0^2 - k^2\right)^2$$

where σ is the growth rate of the perturbation. The solution to this equation is

$$A(t) = A(0)e^{\sigma(k)t}$$

Class I: Linear stability analysis of the uniform state

The solution $u_s=0$ is stable if $\sigma(k)<0$ for all wavenumbers k. Plotting the growth rate $\sigma=\sigma(k)$ for various λ values we find:



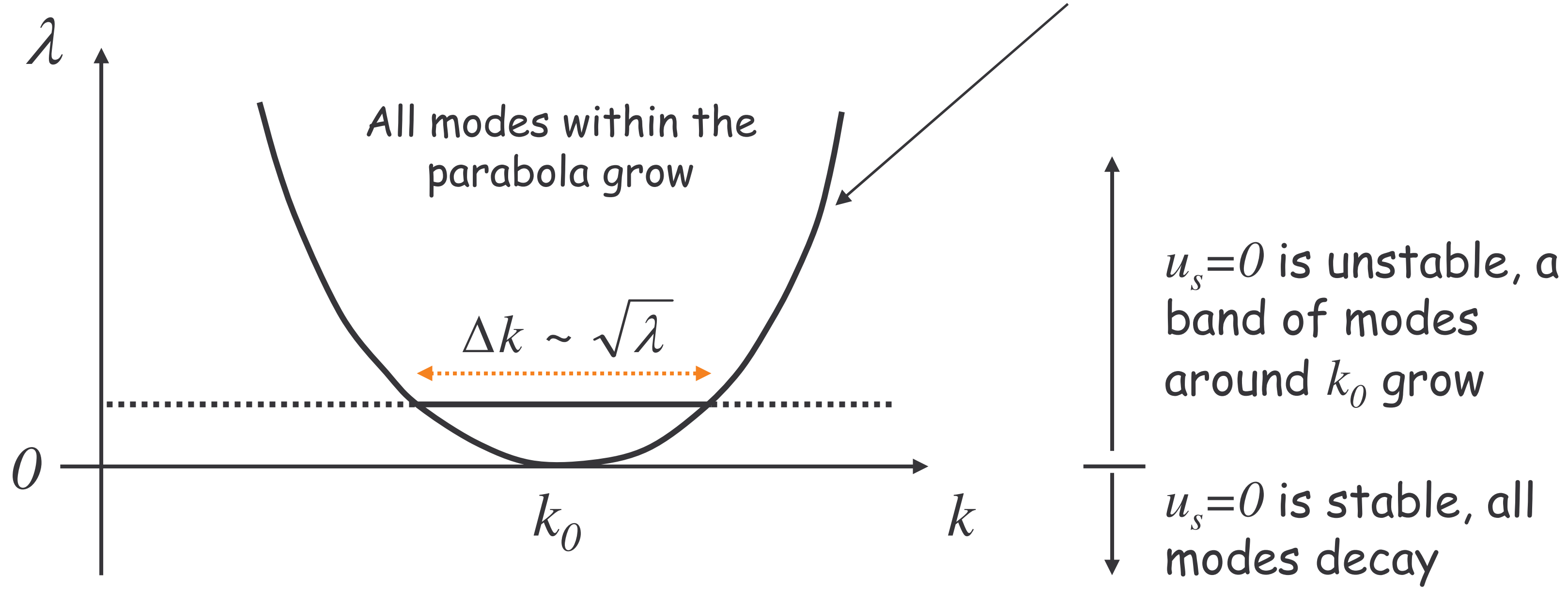
$\sigma<0$ for all k $u_s=0$ is stable	$\sigma(k_0)=0$, $u_s=0$ is marginally stable	$\sigma(k)>0$ for a band of modes, $u_s=0$ is unstable

$\Rightarrow$ The solution $u_s=0$ loses stability at $\lambda = 0$ to the growth of a mode with wavenumber k_0 which breaks the translational symmetry of the system.

This is a major mechanism of symmetry breaking in nonequilibrium systems

Class I: Linear stability analysis of the uniform state

Another way to describe the instability is by plotting a neutral stability curve, by setting $\sigma = 0 \quad \Rightarrow \quad \lambda = \left(k_0^2 - k^2\right)^2$



Linear stability analysis provides information about instability thresholds, $\lambda = 0$ in the case, and about the modes that grow, a finite-wavenumber mode k_0 in this case.

It does not provide information about the new state that the system evolves to because such information can only be derived by considering large deviations from the original state $u_s=0$.

Class I: Nonlinear analysis, amplitude equations

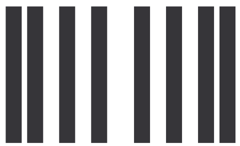
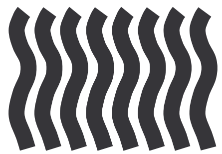
Going back to the SH model (now in two space dimensions)

$$u_t = \lambda u - u^3 - \left(\nabla^2 + k_0^2\right)^2 u \qquad -\infty < x < \infty, \quad -\infty < y < \infty$$

we consider approximate solutions of the form

$$u(x,t) \cong A(x,y,t)\exp(ik_0 x) + c.c. \qquad 0 < |A| << 1$$

where the amplitude A is assumed to be small but finite. We further assume that A varies weakly in space and in time. Such a solution describes weak modulations of a stripe pattern, either

longitudinal  or transverse  or both.

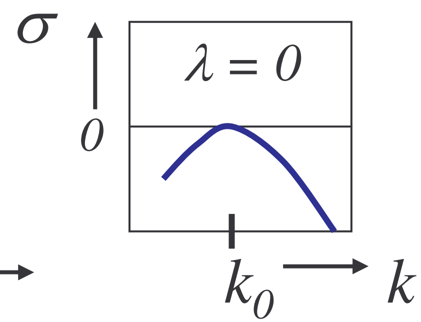
Using various mathematical methods, e.g. multiple time scales, a nonlinear equation for the amplitude A can be derived from the original SH equation:

Class I: Nonlinear analysis, amplitude equations

The equation, known as the Newell-Whitehead-Segel (NWS) equation, reads:

$$\frac{\partial A}{\partial t} = \lambda A + \left(2k_0 \frac{\partial}{\partial x} - i\frac{\partial^2}{\partial y^2}\right)^2 A - 3|A|^2 A$$

Equations of this kind are called "amplitude equations". They have universal forms that are determined by the types of instabilities the systems go through. Any system that goes through a finite-k instability of the form →



will be described close to the instability point by a NWS equation.

More generally, different systems that experience the same type of instability are described by the same amplitude equation and will behave similarly near the instability point.

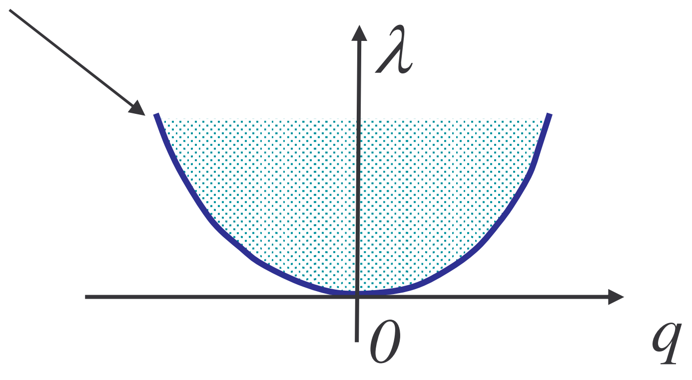
This explains the universal nature of many of the patterns observed in nature.

The amplitude equation has periodic solutions of the form:

$$A = A_0 \exp(iqx), \qquad A_0 = \sqrt{\left(\lambda - 4k_0^2 q^2\right)/3}$$

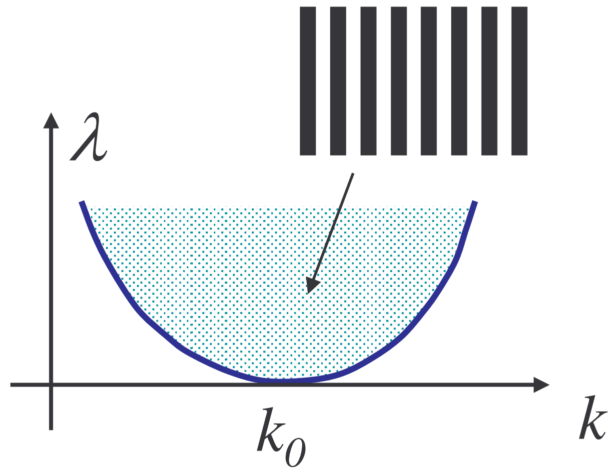
which exist within the parabola

$\lambda = 4k_0^2 q^2$



or

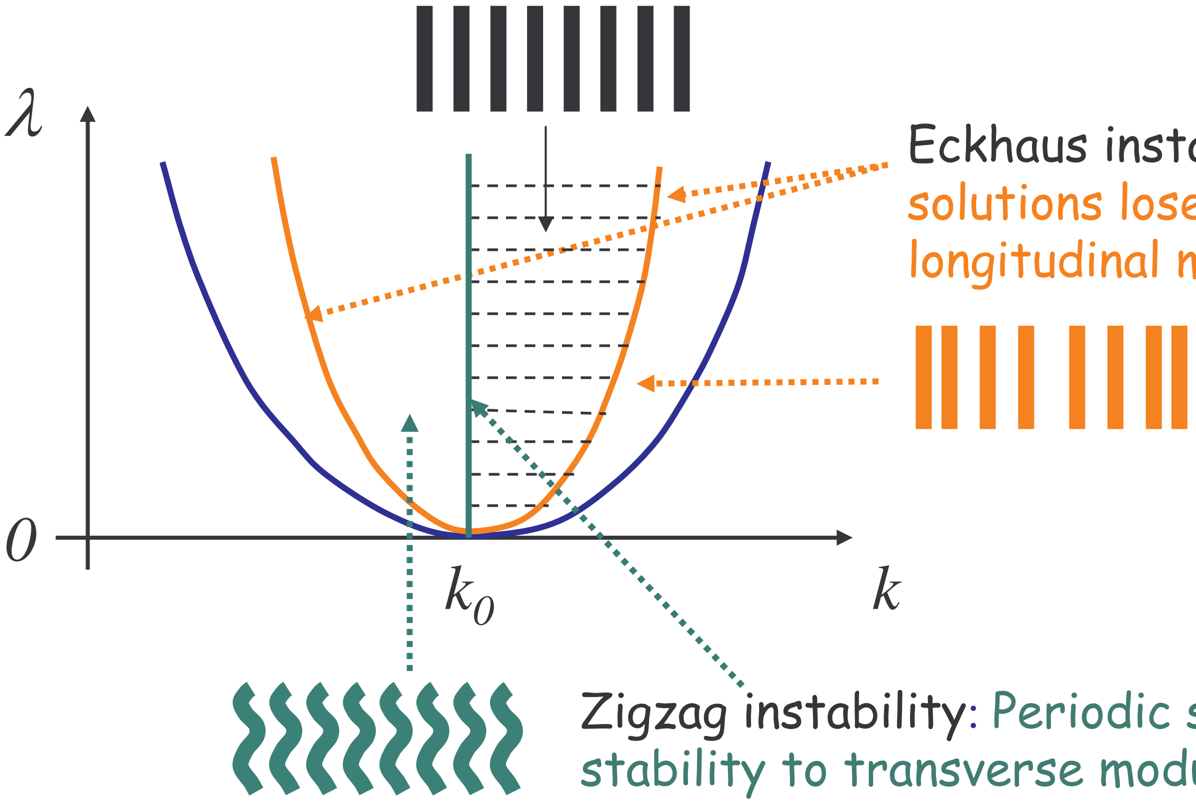
$k = k_0 \pm |q|$



Close to the instability point ($0<\lambda<<1$) it coincides with the parabola $\lambda = \left(k_0^2 - k^2\right)^2$ for the growing modes obtained from the linear stability analysis. This does not imply that all modes that grow from the zero state evolve to periodic solutions – some of them may be unstable.

The amplitude equation can be used to study the linear stability of periodic solutions which exist inside the blue parabola. Such an analysis gives the following boundaries of longitudinal and transverse instabilities:

Periodic solutions are stable:



Eckhaus instability: Periodic solutions lose stability to longitudinal modulations

Zigzag instability: Periodic solutions lose stability to transverse modulations

Read more in Cross & Hohenberg, Rev. Mod. Phys. 1993.

Class I: Summary

Instabilities may break translational symmetry and give rise to spatial patterns even when the system and the forces it is subjected to are uniform.

The spatial patterns induced by instabilities are universal; the same patterns appear in different systems that go through the same instability.

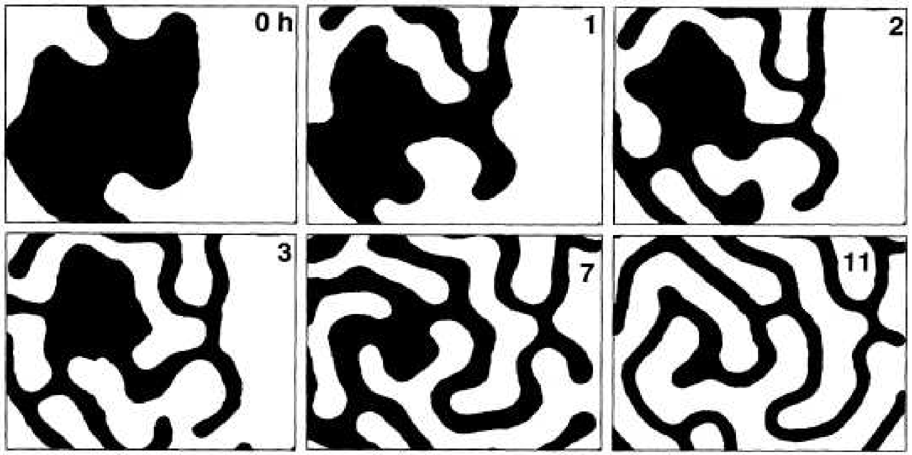
The dynamics close to instability points are described by universal equations – the so called amplitude equations , e.g. NWS equation for a system close to a stationary finite- wavenumber instability.

Amplitude equations are useful for studying secondary instabilities, such as Eckhaus and zigzag.

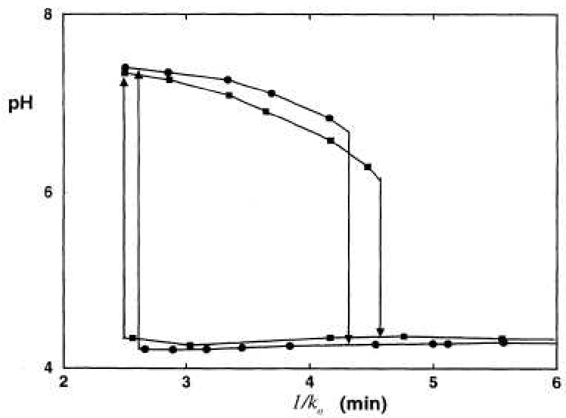
Class II: Multi-stable systems

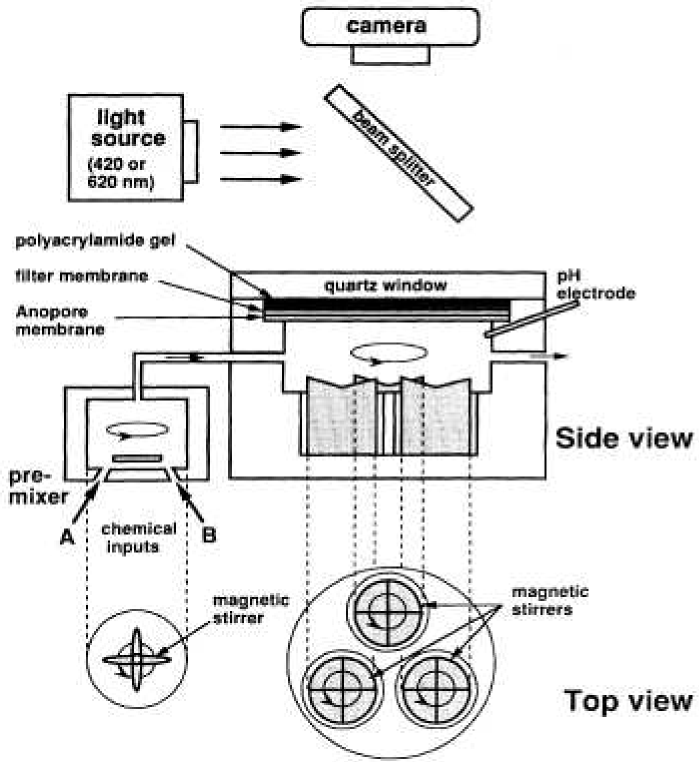
Multistability of uniform states can result in asymptotic patterns even if all the uniform states are linearly stable.

The FIS (ferrocyanide-iodate-sulfite) reaction: A bistable system of high (white) and low (black) pH. Develops patterns by a transverse front instability (class II).



Bistability and hysteresis in the FIS reaction





Lee & Swinney, PRE (1995)

Class II: Bistable systems

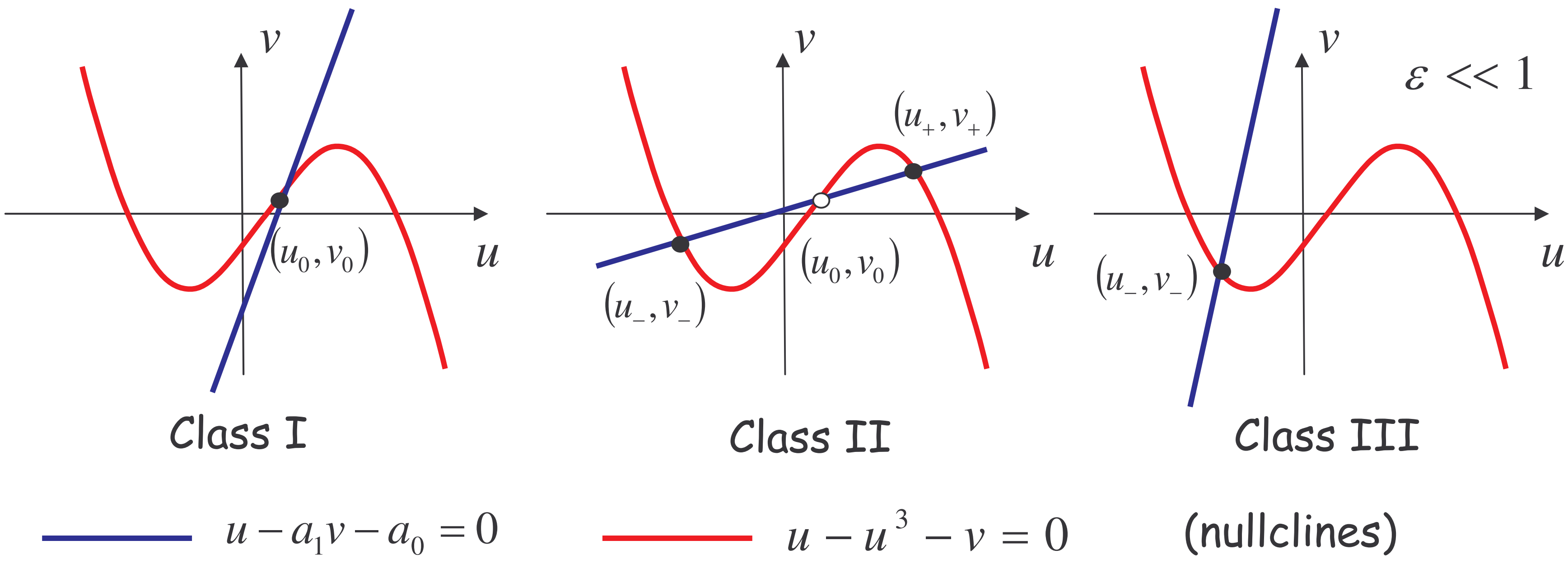
We will study pattern formation phenomena in bistable systems using the FitzHugh-Nagumo (FHN) model:

$$u_t = u - u^3 - v + \nabla^2 u$$

$$v_t = \varepsilon(u - a_1 v - a_0) + \delta\nabla^2 v$$

This is an example of an Activator-Inhibitor system: u activates the growth of itself and v, while v inhibits the growth of itself and u.

The FHN model captures all 3 classes of pattern forming systems:

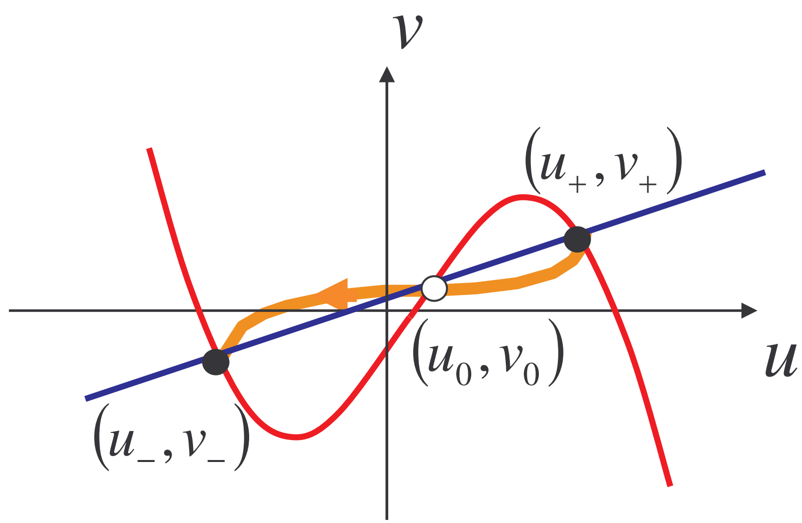


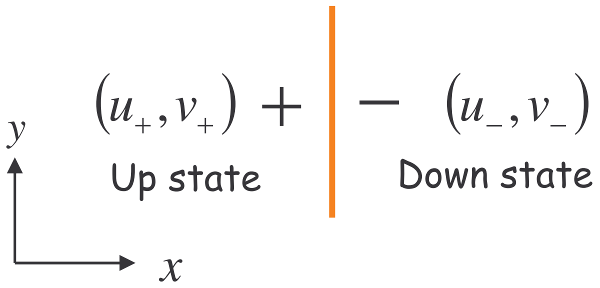
Class II: Bistable systems - fronts

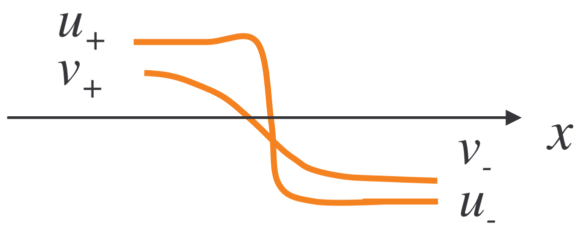
We focus here on class II, corresponding to bistable systems that consist of two stable uniform stationary states: $u=u_-,\ \ v=v_-$ and $u=u_+,\ \ v=v_+$

Such systems support in general front solutions $u(x,t)$ that are bi-asymptotic to the two stable states:

$$u(x,t)\to u_+ \quad v(x,t)\to v_+ \quad as \quad x\to -\infty$$
$$u(x,t)\to u_- \quad v(x,t)\to v_- \quad as \quad x\to +\infty$$







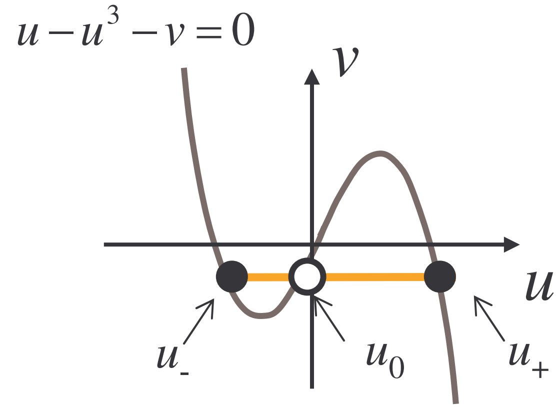
We will see in the following that pattern formation phenomena in bistable systems strongly depend on the properties of these fronts solutions.

Class II: Bistable systems - fronts

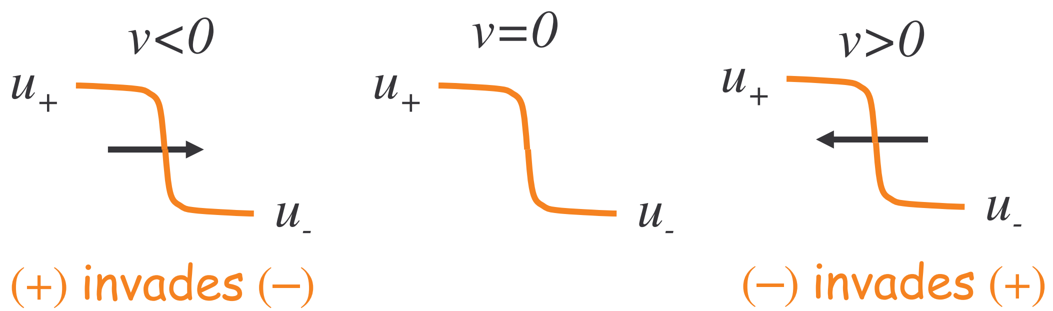
Let's begin with a simpler system, assuming v is constant,

$$u_t = u - u^3 - v + u_{xx} \qquad [\Rightarrow \text{ a gradient system }]$$

For a certain range of v there are two stable stationary uniform solutions, $u_+(v)$ and $u_-(v)$, and an unstable solution $u_0(v)$ in between: →



Front solutions that are biasymptotic to the two stable states propagate at constant speeds, which depend on the value of v:

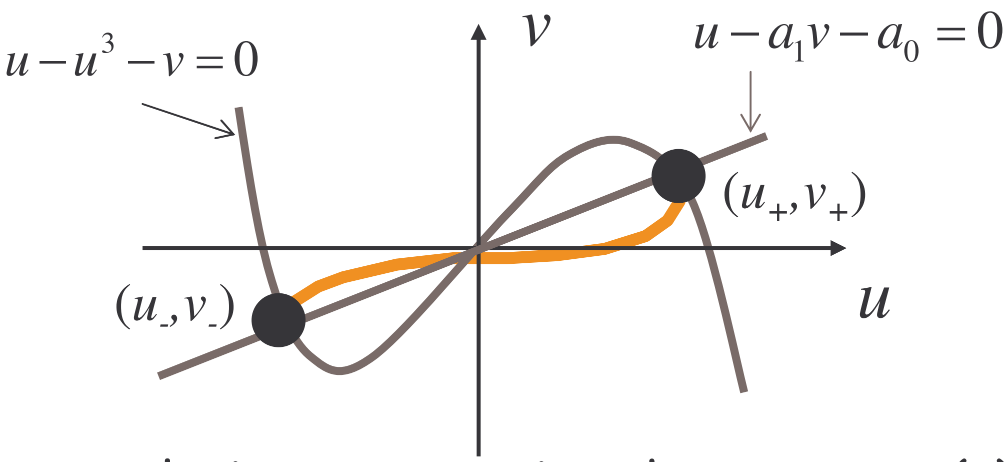


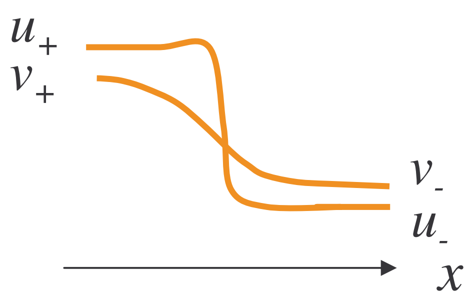
[Fronts propagate so as to minimize a Lyapunov functional]

Note: front speed is uniquely determined by the value of v.

Class II: Bistable systems – front instabilities

The situation becomes completely different when the equation for the inhibitor v is added:

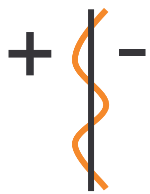




Front solutions connecting the up-state (+) (at $x\to -\infty$) to the down-state (-) (at $x\to -\infty$) can themselves undergo instabilities.

Two front instabilities:

A transverse instability

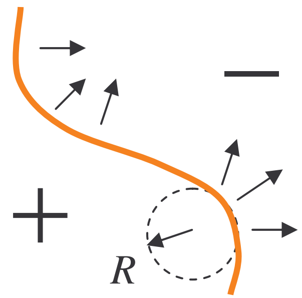


A longitudinal instability

Class II: Bistable systems – front instabilities

To understand these instabilities and their effects on pattern formation we first need to understand how the curvature of a front κ, affects its normal velocity, C (velocity in a direction normal to the front line).

The curvature affects the diffusion of the activator and the inhibitor and thereby the front velocity C:



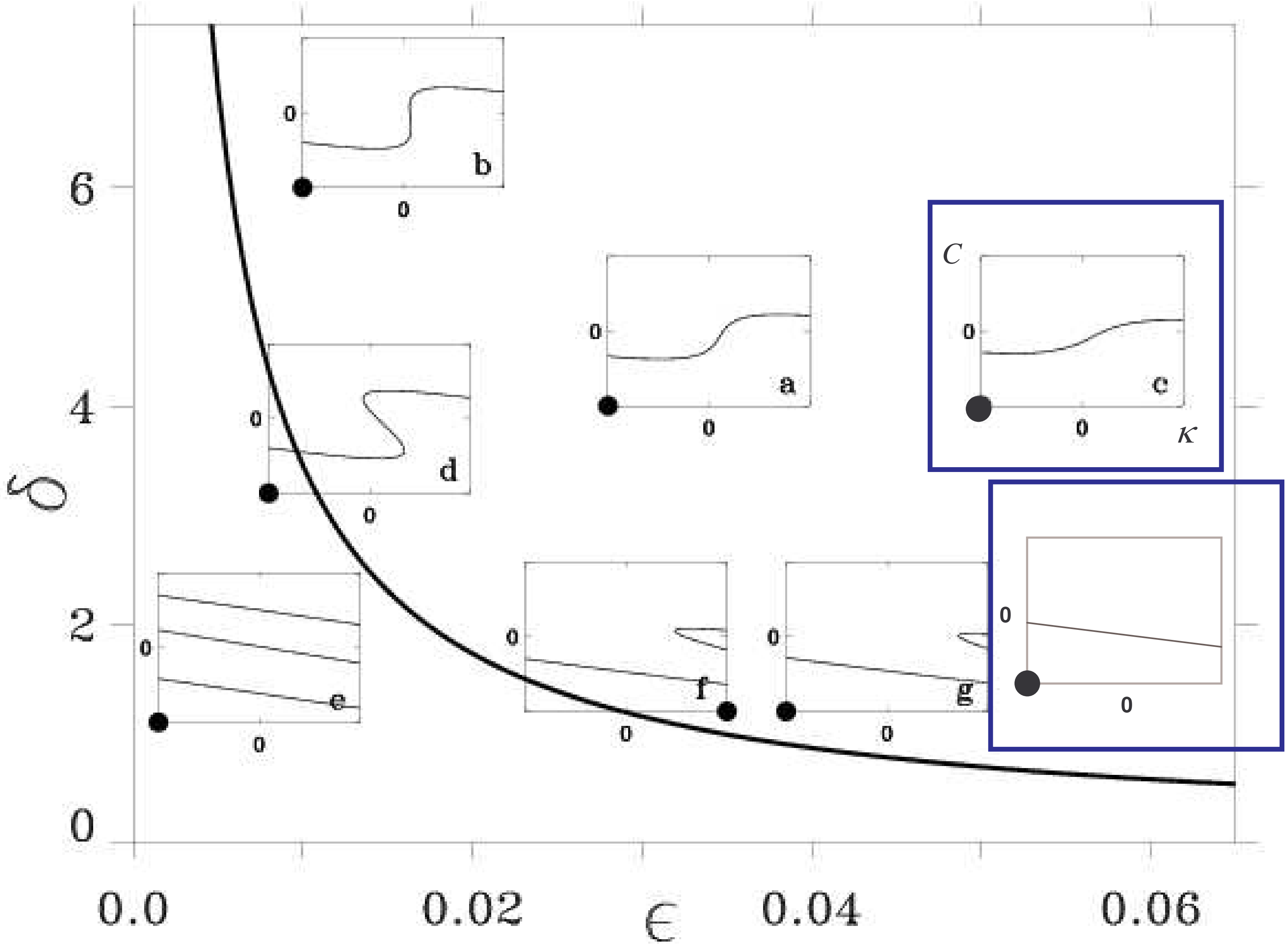
$$C = C(\kappa) \qquad \kappa = 1/R$$

Radius of curvature

The actual form of the velocity-curvature relation will be affected by the parameters that control these diffusion processes: δ and ε.

Using a singular perturbation analysis in the range $\varepsilon/\delta << 1$, which is often met in actual systems, velocity-curvature relationships can be derived.

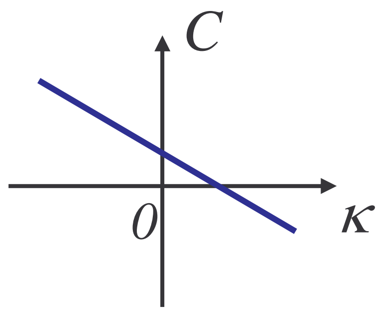
Class II: Bistable systems – front instabilities



Hagberg & Meron, Chaos (1994), Nonlinearity (1994)

Class II: Bistable systems - stationary patterns

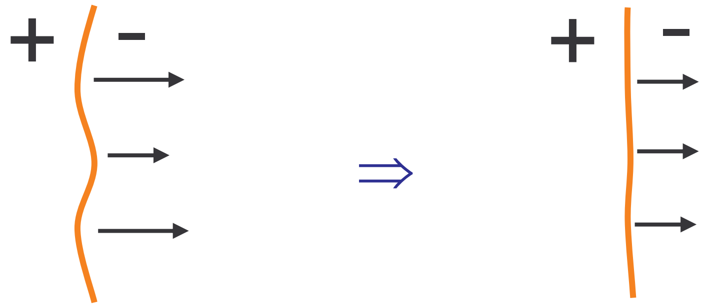
Front stability to transverse perturbations



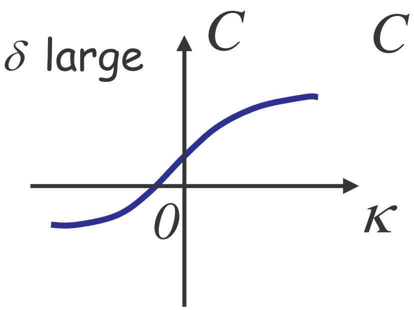
$$C = C_0 - D\kappa$$

Front is stable

(δ small - slow v diffusion)



δ large

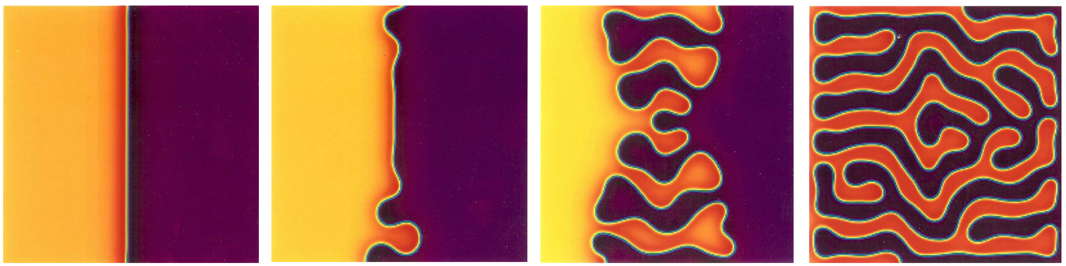


$$C = C_0 + D_1\kappa - D_3\kappa^3$$

Front is unstable

(δ large - fast v diffusion)

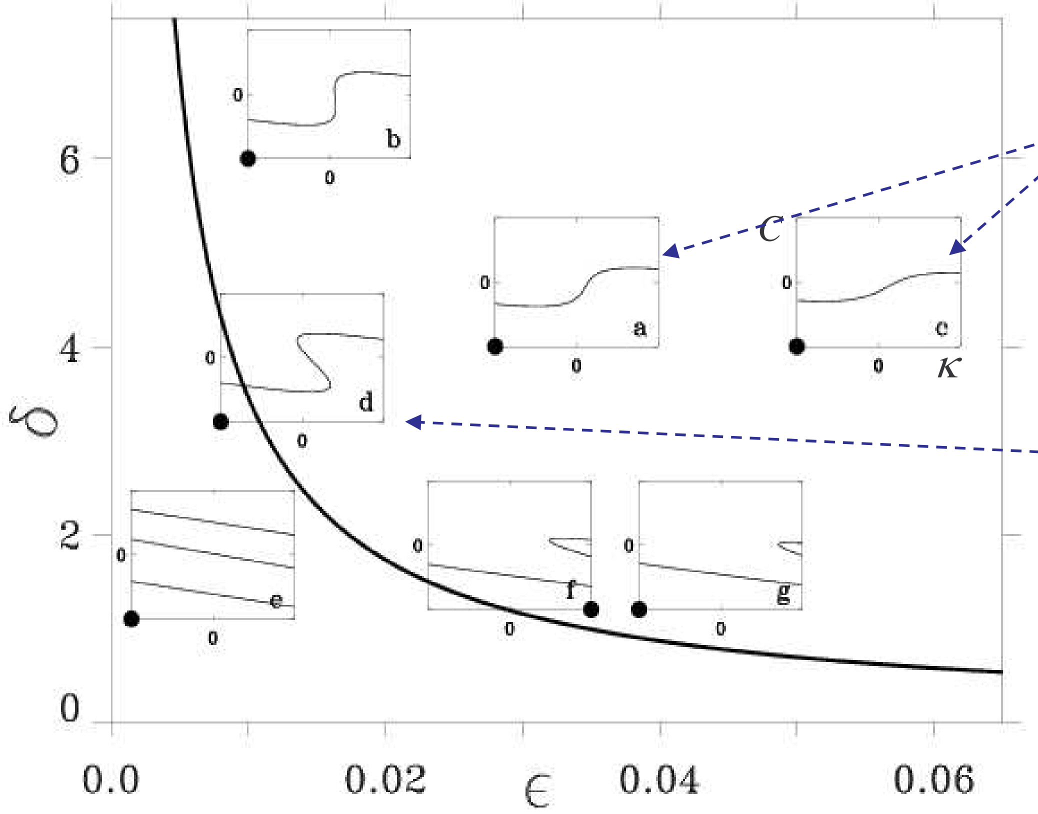
Under these conditions stationary labyrinthine patterns develop:



Note: domains do not merge due to fast v diffusion

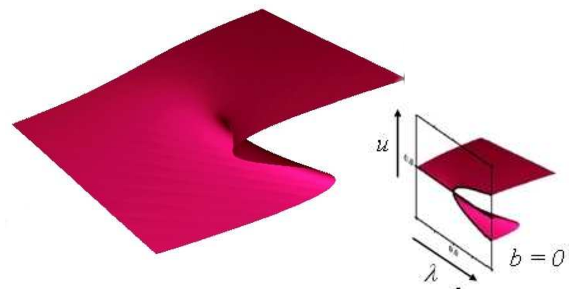
Mimura 2000

Front stability to longitudinal perturbations



Stationary labyrinthine patterns are found here where δ is large (fast v diffusion) and ε not too small.

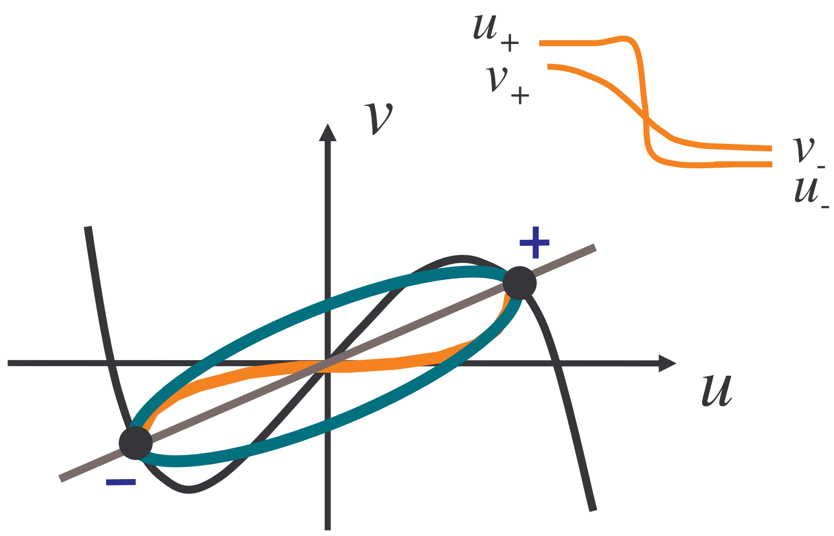
Reminiscent of the behavior near a cusp singularity or pitchfork bifurcation



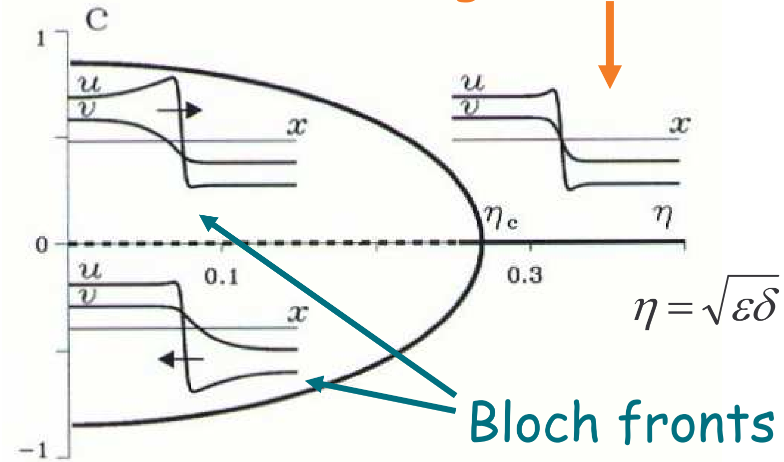
The solid line is the threshold of a longitudinal front instability of the pitchfork type commonly referred to as the "Non-equilibrium Ising-Bloch" (NIB) bifurcation.

Class II: Bistable systems – traveling waves

The Non-equilibrium Ising-Bloch (NIB) bifurcation:



Ising front (Coullet)

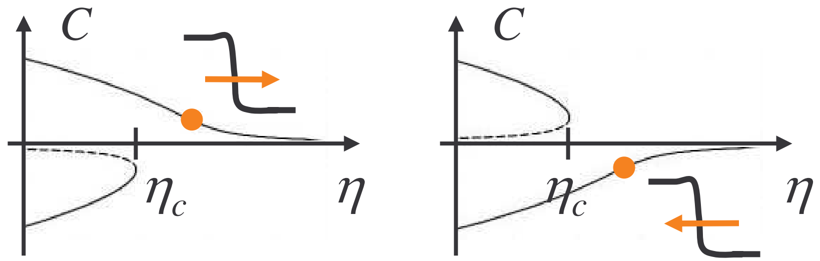


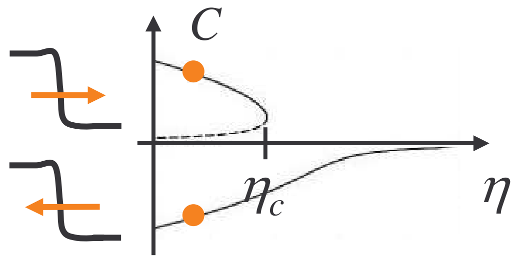
Bloch fronts

Ikeda, Mimura, Nishiura, 1989
Hagberg and Meron 1994

In Ising regime ($\eta > \eta_c$) fronts have unique direction of propagation

In Bloch regime ($\eta < \eta_c$) fronts propagate in opposite directions





(+) invades (–) **or** (–) invades (+)

(+) invades (–) **and** (–) invades (+)

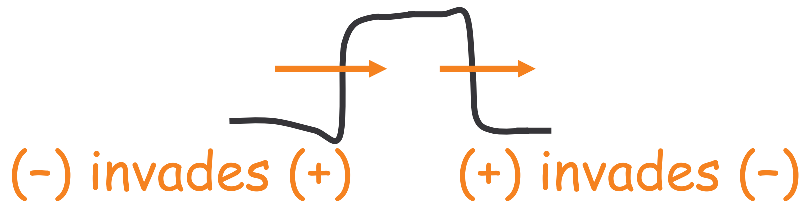
What are the implications for pattern formation?

1. Onset of traveling waves

The NIB designates a transition from stationary or uniform patterns to traveling waves

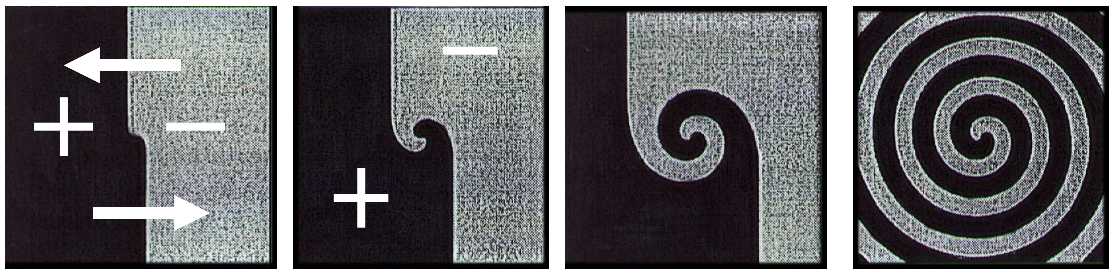
Ising regime: domains can expand, shrink or become stationary

Bloch regime: domains can travel



2. Spiral waves:

Point e



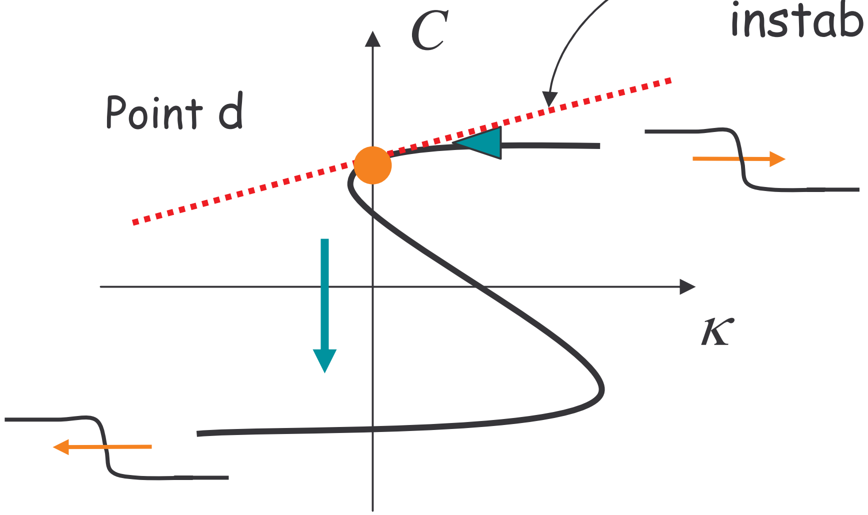
Gilli and Frisch (1995)

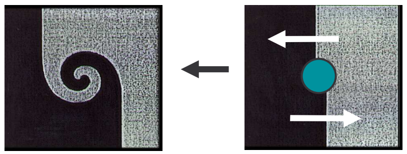
Close to the NIB bifurcation – more complex dynamics

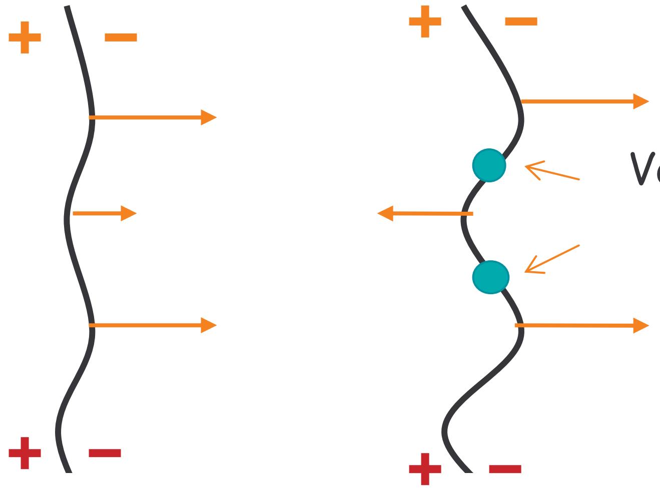
Hagberg & Meron, Chaos (1994), PRL (1994); Elphick, Hagberg & Meron PRE (1995).

3. Bloch-front turbulence:

Positive slope at $\kappa=0$ implies transverse instability





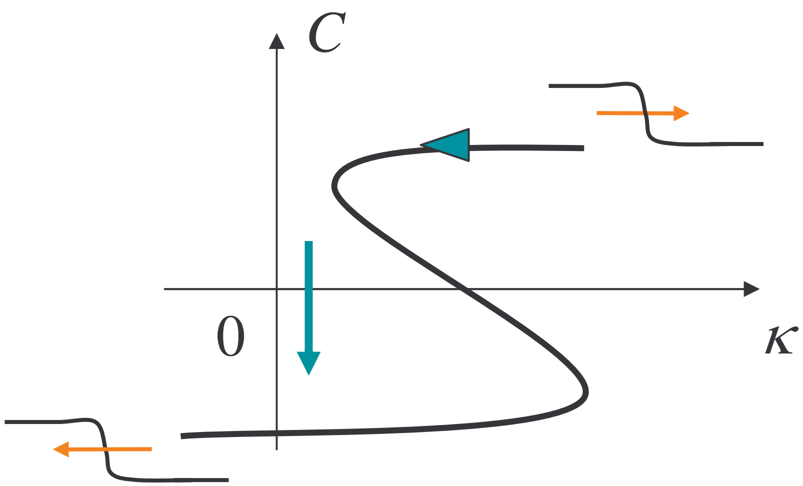


Growing transverse perturbations

After spiral vortex nucleation

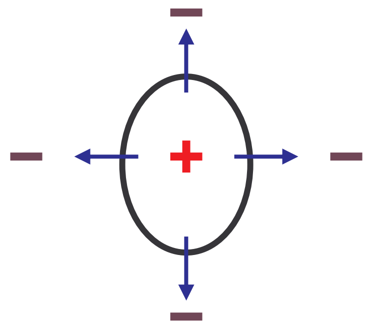
Simulations of the FHN model close to the front bifurcation

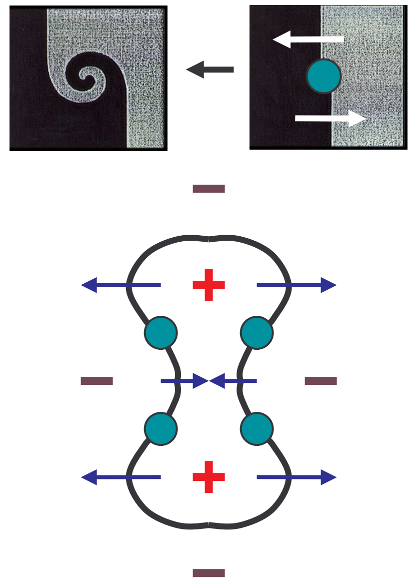
4. Spot splitting:



Point b

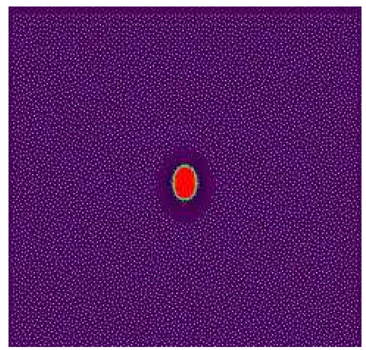
Initial spot





Local front reversals induce nucleation of spiral-vortex pairs

Simulations of the FHN model in the Ising regime close to the front bifurcation



Up state (+) Down state (-)

Hagberg and Meron, PRL (1994), Nonlinearity (1994), PRL (1997), Physica D (1998)

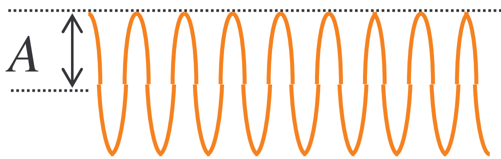
Class II: Summary

Bistability of states allows for domain patterns, stationary or traveling patterns, consisting of alternate domains of the different stable states.

Domain patterns are strongly affected by front instabilities:

- A transverse front instability that leads to stationary labyrinthine patterns.
- A longitudinal instability – the NIB bifurcation – that designates the onset of traveling waves.
- Coupling of the two instabilities leads to spiral turbulence, spot splitting, etc.

The multistable states need not be stationary and uniform - periodic states in space or time, can be reduced to stationary uniform states using amplitude equations.



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